Exponents, Radicals, and Polynomials

Unit Overview
In this unit you will explore multiplicative patterns and representations of nonlinear data. Exponential growth and decay will be the basis for studying exponential functions. You will investigate the properties of powers and radical expressions. You will also perform operations with radical and rational expressions.

Key Terms
As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Math Terms
- radical expression
- principal square root
- negative square root
- cube root
- rationalize
- tree diagram
- geometric sequence
- common ratio
- arithmetic sequence
- recursive formula
- exponential growth
- exponential function
- exponential decay
- compound interest
- exponential regression
- term
- polynomial
- coefficient
- constant term
- degree of a term
- degree of a polynomial
- standard form of a polynomial
- descending order
- leading coefficient
- monomial
- binomial
- trinomial
- like terms
- difference of two squares
- square of a binomial
- greatest common factor of a polynomial
- perfect square trinomial
- rational expression

ESSENTIAL QUESTIONS
- How do multiplicative and exponential patterns model the physical world?
- How are adding and multiplying polynomial expressions different from each other?

EMBEDDED ASSESSMENTS
This unit has four embedded assessments, following Activities 21, 23, 25, and 28. They will give you an opportunity to demonstrate what you have learned.

Embedded Assessment 1: Exponents, Radicals, and Geometric Sequences p. 323
Embedded Assessment 2: Exponential Functions p. 353
Embedded Assessment 3: Polynomial Operations p. 383
Embedded Assessment 4: Factoring and Simplifying Rational Expressions p. 419

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Getting Ready

Write your answers on notebook paper. Show your work.

1. Find the greatest common factor of 36 and 54.
2. List all the factors of 90.
3. Which of the following is equivalent to $39 \cdot 26 + 39 \cdot 13$?
   A. $13^9$
   B. $13^4 \cdot 14$
   C. $13^2 \cdot 3^2 \cdot 2$
   D. $13^3 \cdot 3^2$
4. Identify the coefficient, base, and exponent of $4x^5$.
5. Explain two ways to evaluate $15(90 - 3)$.
6. Complete the following table to create a linear relationship.

<table>
<thead>
<tr>
<th>x</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>3</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Graph the function described in the table in Item 6.

8. Use ratios to model the following:
   a. 7.5
   b. Caleb receives 341 of the 436 votes cast for class president.
   
   Students in Mr. Bulluck’s Class
   
<table>
<thead>
<tr>
<th>Girls</th>
<th>Boys</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>19</td>
</tr>
</tbody>
</table>
   
   c. girls to boys
   d. boys to total class members

9. Tell whether each number is rational or irrational.
   a. $\sqrt{25}$
   b. $\frac{4}{3}$
   c. 2.16
   d. $\pi$

10. Calculate.
    a. $\frac{1}{2} + \frac{3}{8}$
    b. $\frac{5}{12} - \frac{1}{3}$
    c. $\frac{2}{2} \cdot \frac{2}{5}$
    d. $\frac{5}{8} \div \frac{3}{4}$
Learning Targets:
• Develop basic exponent properties.
• Simplify expressions involving exponents.

SUGGESTED LEARNING STRATEGIES: Create Representations, Predict and Confirm, Look for a Pattern, Think-Pair-Share, Discussion Groups, Sharing and Responding

An iceberg is a large piece of freshwater ice that has broken off from a glacier or ice shelf and is floating in open seawater. Icebergs are classified by size. The smallest sized iceberg is called a “growler.”

A growler was found floating in the ocean just off the shore of Greenland. Its volume above water was approximately 27 cubic meters.

1. **Reason quantitatively.** Two icebergs float near this growler. One iceberg’s volume is $3^4$ times greater than the growler. The second iceberg’s volume is $2^8$ times greater than the growler. Which iceberg has the larger volume? Explain.

2. What is the meaning of $3^4$ and $2^8$? Why do you think exponents are used when writing numbers?

3. Suppose the original growler’s volume under the water is 9 times the volume above. How much of its ice is below the surface?

4. Write your solution to Item 3 using powers. Complete the equation below. Write the missing terms as a power of 3.

   
   \[ \text{volume above water} \cdot 3^2 = \text{volume below the surface} \]

   \[ \square \cdot 3^2 = \square \]

5. Look at the equation you completed for Item 4. What relationship do you notice between the exponents on the left side of the equation and the exponent on the right?
6. Use the table below to help verify the pattern you noticed in Item 5. First write each product in the table in expanded form. Then express the product as a single power of the given base. The first one has been done for you.

<table>
<thead>
<tr>
<th>Original Product</th>
<th>Expanded Form</th>
<th>Single Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^2 \cdot 2^4$</td>
<td>$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$</td>
<td>$2^6$</td>
</tr>
<tr>
<td>$5^3 \cdot 5^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^4 \cdot x^7$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a^6 \cdot a^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. **Express regularity in repeated reasoning.** Based on the pattern you observed in the table in Item 6, write the missing exponent in the box below to complete the **Product of Powers Property** for exponents.

\[ a^m \cdot a^n = a^{m+n} \]

8. Use the Product of Powers Property to write $x^{\frac{3}{4}} \cdot x^{\frac{5}{4}}$ as a single power.

9. The density of an iceberg is determined by dividing its mass by its volume. Suppose a growler had a mass of 59,049 kg and a volume of 81 cubic meters. Compute the density of the iceberg.

10. Write your solution to Item 9 using powers of 9.

\[ \frac{\text{Mass}}{\text{Volume}} = \text{Density} \]

11. What pattern do you notice in the equation you completed for Item 10?
12. Use the table to help verify the patterns you noticed in Item 11. First write each quotient in the table below in expanded form. Then express the quotient as a single power of the given base. The first one has been done for you.

<table>
<thead>
<tr>
<th>Original Quotient</th>
<th>Expanded Form</th>
<th>Single Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{x^5}{x^2} )</td>
<td>( \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = x \cdot x \cdot x \cdot x \cdot x )</td>
<td>( x^3 )</td>
</tr>
<tr>
<td>( \frac{y^8}{y^6} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{a^3}{a^{-1}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{x^7}{x^3} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

13. Based on the pattern you observed in Item 12, write the missing exponent in the box below to complete the Quotient of Powers Property for exponents.

\[ \frac{a^m}{a^n} = a^{\square} \text{, where } a \neq 0 \]

14. Use the Quotient of Powers Property to write \( \frac{a^7}{a^3} \) as a single power.

The product and quotient properties of exponents can be used to simplify expressions.

**Example A**

Simplify: \( 2x^5 \cdot 5x^4 \)

**Step 1:** Group powers with the same base.

\( 2x^5 \cdot 5x^4 = 2 \cdot 5 \cdot x^5 \cdot x^4 \)

**Step 2:** Product of Powers Property

\( = 10x^5 \cdot x^4 \)

**Step 3:** Simplify the exponent.

\( = 10x^9 \)

**Solution:** \( 2x^5 \cdot 5x^4 = 10x^9 \)
Example B

Simplify: \(\frac{2x^5y^4}{xy^2}\)

Step 1: Group powers with the same base.  
\[
\frac{2x^5y^4}{xy^2} = 2 \cdot \frac{x^5}{x} \cdot \frac{y^4}{y^2} = 2x^{5-1} \cdot y^{4-2}
\]

Step 2: Quotient of Powers Property

Step 3: Simplify the exponents.

Solution: \(\frac{2x^5y^4}{xy^2} = 2x^4y^2\)

Try These A–B

Simplify each expression.

a. \((4xy^4)(-2x^2y^5)\)

b. \(\frac{2a^2b^5c}{4ab^2c}\)

c. \(\frac{6y^3}{18x} \cdot 2xy\)

Check Your Understanding

15. Simplify \(3yz^2 \cdot 5y^2z\).

16. Simplify \(\frac{21f^2g^7}{7f^3g^3}\).

17. A growler has a mass of 243 kg and a volume of 27 cubic meters. Compute the density of the iceberg by completing the following. Write your answer using powers of 3. \(\frac{3^5}{3^3} = \)

LESSON 19-1 PRACTICE

18. Which expression has the greater value? Explain your reasoning.

a. \(2^3 \cdot 2^5\)

b. \(\frac{4^7}{4^3}\)

19. The mass of an object is \(x^{15}\) grams. Its volume is \(x^9\) cm\(^3\). What is the object's density?

20. The density of an object is \(y^{10}\) grams/cm\(^3\). Its volume is \(y^4\) cm\(^3\). What is the object’s mass?

21. Simplify the expression \(\frac{(3x)^\frac{1}{3} \cdot (3x)^\frac{7}{3}}{(3x)^\frac{2}{3}}\).

22. Make sense of problems. Tanika asks Toby to multiply the expression \(8^7 \cdot 8^3 \cdot 8^2\). Toby says he doesn’t know how to do it, because he believes the Product of Powers Property works with only two exponential terms, and this problem has three terms. Explain how Toby could use the Product of Powers Property with three exponential terms.
Learning Targets:

- Understand what is meant by negative and zero powers.
- Simplify expressions involving exponents.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Discussion Groups, Sharing and Responding, Think-Pair-Share, Close Reading, Note Taking

1. **Attend to precision.** Write each quotient in expanded form and simplify it. Then apply the Quotient of Powers Property. The first one has been done for you.

<table>
<thead>
<tr>
<th>Original Quotient</th>
<th>Expanded Form</th>
<th>Single Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2^5}{2^8} )</td>
<td>( \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2^3} )</td>
<td>( 2^{5-8} = 2^{-3} )</td>
</tr>
<tr>
<td>( \frac{5^3}{5^6} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{a^3}{a^8} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{x^4}{x^{10}} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Based on the pattern you observed in Item 1, write the missing exponent in the box below to complete the **Negative Power Property** for exponents.

\[ \frac{1}{a^n} = a^{-n}, \text{ where } a \neq 0 \]

3. Write each quotient in expanded form and simplify it. Then apply the Quotient of Powers Property. The first one has been done for you.

<table>
<thead>
<tr>
<th>Original Quotient</th>
<th>Expanded Form</th>
<th>Single Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2^4}{2^4} )</td>
<td>( \frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2} = 1 )</td>
<td>( 2^{4-4} = 2^0 )</td>
</tr>
<tr>
<td>( \frac{5^6}{5^5} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{a^3}{a^3} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Lesson 19-2
Negative and Zero Powers

4. Based on the pattern you observed in Item 3, fill in the box below to complete the Zero Power Property of exponents.

\[ a^0 = \boxed{1}, \text{ where } a \neq 0 \]

5. Use the properties of exponents to evaluate the following expressions.
   a. \[ 2^{-3} \]
   b. \[ \frac{10^2}{10^{-2}} \]
   c. \[ 3^{-2} \cdot 5^0 \]
   d. \[ (-3.75)^0 \]

When evaluating and simplifying expressions, you can apply the properties of exponents and then write the answer without negative or zero powers.

**Example A**

Simplify \(5x^{-2}yz^0 \cdot \frac{3x^4}{y^4}\) and write without negative powers.

**Step 1:** Commutative Property

\[ 5x^{-2}yz^0 \cdot \frac{3x^4}{y^4} = 5 \cdot 3 \cdot x^{-2} \cdot x^4 \cdot y^1 \cdot y^{-4} \cdot z^0 \]

**Step 2:** Apply the exponent rules.

\[ = 5 \cdot 3 \cdot x^{-2+4} \cdot y^{1-4} \cdot z^0 \]

**Step 3:** Simplify the exponents.

\[ = 15 \cdot x^2 \cdot y^{-3} \cdot 1 \]

**Step 4:** Write without negative exponents.

\[ = \frac{15x^2}{y^3} \]

**Solution:** \(5x^{-2}yz^0 \cdot \frac{3x^4}{y^4} = \frac{15x^2}{y^3}\)

**Try These A**

Simplify and write without negative powers.

a. \(2a^2b^{-3} \cdot 5ab\)
   b. \(\frac{10x^2y^{-4}}{5x^{-3}y^{-1}}\)
   c. \((-3xy^{-5})^0\)
Lesson 19-2
Negative and Zero Powers

Check Your Understanding

Simplify each expression. Write your answer without negative exponents.

6. \((z)^{-3}\)
7. \(12(xyz)^0\)
8. \(\frac{6^4}{6^{-2}}\)
9. \(2^3 \cdot 2^{-6}\)
10. \(\frac{4x^{-2}}{x^3}\)
11. \(\frac{-5}{(ab)^0}\)

LESSON 19-2 PRACTICE

12. For what value of \(v\) is \(a^v = 1\), if \(a \neq 0\)?
13. For what value of \(w\) is \(b^{-w} = \frac{1}{b^2}\), if \(b \neq 0\)?
14. For what value of \(y\) is \(\frac{3^3}{3^y} = \frac{1}{9}\)?
15. For what value of \(z\) is \(5^8 \cdot 5^z = 1\)?
16. Determine the values of \(n\) and \(m\) that would make the equation \(7^n \cdot 7^m = 1\) a true statement. Assume that \(n \neq m\).
17. For what value of \(x\) is \(\frac{3^x \cdot 2^3}{3^4} = \frac{4}{3}\)?
18. Reason abstractly. What is the value of \(2^0 \cdot 3^0 \cdot 4^0 \cdot 5^0\)? What is the value of any multiplication problem in which all of the factors are raised to a power of 0? Explain.
Learning Targets:

- Simplify expressions involving exponents.

SUGGESTED LEARNING STRATEGIES: Note Taking, Look for a Pattern, Create Representations, Think-Pair-Share, Sharing and Responding, Close Reading

1. Write each expression in expanded form. Then write the expression using a single exponent with the given base. The first one has been done for you.

<table>
<thead>
<tr>
<th>Original Expression</th>
<th>Expanded Form</th>
<th>Single Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2^2)^4$</td>
<td>$2^2 \cdot 2^2 \cdot 2^2 \cdot 2^2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$</td>
<td>$2^8$</td>
</tr>
<tr>
<td>$(5^5)^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(x^3)^4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Based on the pattern you observed in Item 1, write the missing exponent in the box below to complete the Power of a Power Property for exponents.

$$(a^m)^n = a^{m \cdot n}$$

3. Use the Power of a Power Property to write $\left(\frac{6}{x^5}\right)^{25}$ as a single power.

4. Write each expression in expanded form and group like terms. Then write the expression as a product of powers. The first one has been done for you.

<table>
<thead>
<tr>
<th>Original Expression</th>
<th>Expanded Form</th>
<th>Product of Powers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2x)^4$</td>
<td>$2x \cdot 2x \cdot 2x \cdot 2x = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x$</td>
<td>$2^4x^4$</td>
</tr>
<tr>
<td>$(-4a)^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(x^3y^2)^4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5. Based on the pattern you observed in Item 4, write the missing exponents in the boxes below to complete the **Power of a Product Property** for exponents.

\[(ab)^m = a \square \cdot b \square\]

6. Use the Power of a Product Property to write \(\left(c^2 d^4\right)^8\) as a product of powers.

7. **Make use of structure.** Use the patterns you have seen. Predict and write the missing exponents in the boxes below to complete the **Power of a Quotient Property** for exponents.

\[\left(\frac{a}{b}\right)^m = \frac{a \square}{b \square}, \text{ where } b \neq 0\]

8. Use the Power of a Quotient Property to write \(\left(\frac{x^3}{y^6}\right)^{\frac{1}{3}}\) as a quotient of powers.

You can apply these power properties and the exponent rules you have already learned to simplify expressions.

**Example A**

Simplify \((2x^2y^5)^3 (3x^2)^{-2}\) and write without negative powers.

**Step 1:** Power of a Power Property

\[(2x^2y^5)^3 (3x^2)^{-2} = 2^3x^{2\cdot3}y^{5\cdot3} \cdot 3^{-2} \cdot x^2 \cdot -2\]

**Step 2:** Simplify the exponents and the numerical terms.

\[= 8 \cdot x^6y^{15} \cdot \frac{1}{3^2} \cdot x^{-4}\]

**Step 3:** Commutative Property

\[= 8 \cdot \frac{1}{9} x^6 \cdot x^{-4} y^{15}\]

**Step 4:** Product of Powers Property

\[= \frac{8}{9} x^{6-4} y^{15}\]

**Step 5:** Simplify the exponents.

\[= \frac{8}{9} x^2 y^{15}\]

**Solution:** \((2x^2y^5)^3 (3x^2)^{-2} = \frac{8}{9} x^2 y^{15}\)
Example B
Simplify \( \left( \frac{x^2 y^{-3}}{z} \right)^2 \).

Step 1: Power of a Quotient Property
\[
\left( \frac{x^2 y^{-3}}{z} \right)^2 = \frac{x^{2+2} y^{-3-2}}{z^2}
\]

Step 2: Simplify the exponents.
\[
= \frac{x^4 y^{-6}}{z^2}
\]

Step 3: Negative Power Property
\[
= \frac{x^4}{y^6 z^2}
\]

Solution: \( \left( \frac{x^2 y^{-3}}{z} \right)^2 = \frac{x^4}{y^6 z^2} \)

Try These A–B
Simplify and write without negative powers.

a. \((2x^2 y)^3 (-3xy^3)^2\)  
   b. \(-2ab(5b^2 c)^3\)  
   c. \(\left(\frac{4x}{y^3}\right)^{-2}\)

d. \(\left(\frac{5x}{y^3}\right)^2 \left(\frac{y^3}{10x^2}\right)\)  
   e. \((3xy^{-2})^2(2x^2yz)(6yz)^{-1}\)

Check Your Understanding
Simplify each expression. Write your answer without negative exponents.

9. \((4x^3 y^{-1})^2\)  
10. \(\left(\frac{5x}{y^2}\right)^3\)
11. \((-2a^2 b^{-2} c)^3(3ab^4 c^5)(xyz)^0\)  
12. \((4fg^3)^{-2} (-4fg^3 h)^2(3gh^4)^{-1}\)
13. \(\left(\frac{2ab}{a^2 b^{-2}}\right)^{-3}\)  
14. \(\left(\frac{7nm^3}{3}\right)^{-3}\)

LESSON 19-3 PRACTICE
Simplify.

15. a. \(\left(\frac{2}{3}\right)^2\)  
   b. \(\left(\frac{2}{3}\right)^{-2}\)
16. a. \((3x)^3\)  
   b. \((3x)^{-3}\)
17. a. \((2^5)^4\)  
   b. \((2^5)^{-4}\)
18. Model with mathematics. The formula for the area of a square is \(A = s^2\), where \(s\) is the side length. A square garden has a side length of \(x^4 y\). What is the area of the garden?
**ACTIVITY 19 PRACTICE**

Write your answers on notebook paper. Show your work.

**Lesson 19-1**

For Items 1–5, evaluate the expression. Write your answer without negative powers.

1. \(x^8 \cdot x^7\)
2. \(\frac{6a^{10}b^9}{3ab^3}\)
3. \((6a^2b)(-3ab^3)\)
4. \(\frac{7x^2y^5}{14xy^4}\)
5. \(\frac{2xy^2}{x^2y^3} \cdot \frac{5xy^3}{-30y^{-2}}\)

6. The volume of an iceberg that is below the water line is 2.5 cubic meters. The volume that is above the water line is 2.2 cubic meters. How many times greater is the volume below the water line than above it?
   - A. 2.25
   - B. 2.3
   - C. 2.7
   - D. 2.10

7. A megabyte is equal to 2\(^{20}\) bytes, and a gigabyte is equal to 2\(^{30}\) bytes. How many times larger is a gigabyte than a megabyte?

8. A jackpot is worth 10\(^5\) dollars. The contestant who wins the jackpot has the opportunity to put it all on the line with the single spin of a prize wheel. If the contestant spins the number 7 on the wheel, she will win 10\(^2\) times more money. How many dollars will the contestant win if she risks her prize money and spins a 7?

9. How many earthquakes of magnitude 8 are likely to occur next year?

10. If an earthquake of magnitude 10 occurred last year, how many years will it be before another one of that magnitude is likely to occur?

**Lesson 19-2**

11. Which of the following expressions is not equal to 1?
   - A. \(x^3 \cdot x^{-3}\)
   - B. 1001\(^0\)
   - C. \(\frac{a^2b}{ba^2}\)
   - D. \(\frac{y^2}{y^{-2}}\)

12. Which of the following expressions is equal to \(\frac{y}{x^2}\)?
   - A. \(x^{-2}y^3 \cdot y^{-2}\)
   - B. \(xy^2 \cdot x^{-3}y^{-2}\)
   - C. \(\frac{y^2x}{yx^{-3}}\)
   - D. \(\frac{x^2y}{y^{-2}}\)

Determine whether each statement is always, sometimes, or never true.

13. For \(a \neq 0\), the value of \(a^{-1}\) is positive.

14. If \(n\) is an integer, then \(3^n \cdot 3^{-n}\) equals 1.

15. If \(6^p > 0\), then \(p > 0\).

16. \(4^{-x}\) equals \(\frac{1}{4^x}\).

17. If \(m\) is an integer, then the value of \(2^m\) is negative.
18. For what value of \(a\) is \(w^{a-2} = 1\), if \(w \neq 0\)?

19. For what value of \(b\) is \(p^{b-1} = \frac{1}{p^3}\), if \(p \neq 0\)?

For each of the following, give the value of the expression or state that the expression is undefined.

20. \(x^0\) when \(x = 0\)

21. \(2^{-a}\) when \(a = 0\)

22. \(\frac{1}{x^p}\) when \(x = 0\) and \(p > 0\)

23. \(0^n \cdot 0^{-n}\) when \(n\) is an integer

**Lesson 19-3**

24. The area of a square is given by the formula \(A = s^2\), where \(s\) is the length of the side. What is the area of the square shown?

![Square Diagram]

25. What is the volume of the cube shown?

![Cube Diagram]

26. What is the volume of the cube shown?

![Cube Diagram]

27. The volume of a cube is \(x^{27}\) cubic inches. What expression represents the length of one side of the cube? Justify your reasoning.

Simplify each expression. Write your answer without negative exponents.

28. \((-5x^2y^{-1})^4\)

29. \(\left(\frac{c^2d^{-2}}{e}\right)^5\)

30. \((x^2 y^2 z^{-1})^3 (xyz^4)(x^3 y)\)

31. \((m^2 n^{-5})^0 m^{-7}\)

32. \(\left(\frac{2x^{-2}}{3}\right)\left(\frac{3x}{4}\right)^2\)

33. Which of the following is a true statement about the expression \(a^4 \left(\frac{1}{a}\right)^2\), given that \(a \neq 0\)?
   A. The expression is always equal to 1.
   B. The value of the expression is positive.
   C. If \(a\) is negative, then the value of the expression is also negative.
   D. The expression cannot be simplified any further.

**MATHEMATICAL PRACTICES**

*Construct Viable Arguments and Critique the Reasoning of Others*

34. Alana says that \((ab)^3 \cdot (ab)^4\) is the same as \([ab]^3]^4\). Is Alana correct? Justify your response.
**Learning Targets:**
- Write and simplify radical expressions.
- Understand what is meant by a rational exponent.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Close Reading, Discussion Groups, Sharing and Responding, Note Taking, Think-Pair-Share

The frame of a box kite has four “legs” of equal length and four pairs of crossbars, all of equal length. The legs of the kite form a square base. The crossbars are attached to the legs so that each crossbar is positioned as a diagonal of the square base.

1. **a.** Label the legs of the kite pictured to the right. How many legs are in a kite? How many crossbars?

   **b.** Label the points on the top view where the ends of the crossbars are attached to the legs A, B, C, and D. Begin at the bottom left and go clockwise.

   **c.** Use one color to show the sides of the square and another color to show crossbar AC. What two figures are formed by two sides of the square and one diagonal?

Members of the Windy Hill Science Club are building kites to explore aerodynamic forces. Club members will provide paper, plastic, or lightweight cloth for the covering of their kite. The club will provide the balsa wood for the frames.

2. **Model with mathematics.** The science club advisor has created the chart below to help determine how much balsa wood he needs to buy.
   **a.** For each kite, calculate the exact length of one crossbar that will be needed to stabilize the kite. Use your drawing from Item 1c as a guide for the rectangular base of these box kites.

<table>
<thead>
<tr>
<th>Kite</th>
<th>Dimensions of Base (in feet)</th>
<th>Exact Length of One Crossbar (in feet)</th>
<th>Kite</th>
<th>Dimensions of Base (in feet)</th>
<th>Exact Length of One Crossbar (in feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 by 1</td>
<td></td>
<td>D</td>
<td>1 by 2</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>2 by 2</td>
<td></td>
<td>E</td>
<td>2 by 4</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>3 by 3</td>
<td></td>
<td>F</td>
<td>3 by 6</td>
<td></td>
</tr>
</tbody>
</table>

   **b.** How much wood would you recommend buying for the crossbars of Kite A? Explain your reasoning.
Each amount of wood in the table in Item 2 is a radical expression.

**Radical Expression**
An expression of the form $\sqrt[n]{a}$, where $a$ is the radicand, $\sqrt{}$ is the radical symbol, and $n$ is the root index.

$$\sqrt[n]{a} = b,$$ if $b^n = a.$  

$b$ is the $n$th root of $a$.

Finding the square root of a number or expression is the inverse operation of squaring a number or expression.

$$\sqrt{25} = 5,$$ because $(5)(5) = 25$

$$\sqrt{81} = 9,$$ because $(9)(9) = 81$

$$\sqrt{x^2} = x,$$ because $(\sqrt{x})(\sqrt{x}) = x^2, x \geq 0$

Notice also that $(-5)(-5) = (-5)^2 = 25$. The principal square root of a number is the positive square root value. The expression $\sqrt{25}$ simplifies to 5, the principal square root. The negative square root is the negative root value, so $-\sqrt{25}$ simplifies to $-5$.

To simplify square roots in which the radicand is not a perfect square:

**Step 1:** Write the radicand as a product of numbers, one of which is a perfect square.

**Step 2:** Find the square root of the perfect square.

---

**Example A**
Simplify each expression.

a. $\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}$

b. $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

$$\sqrt{72} = \sqrt{9 \cdot 4 \cdot 2} = (3 \cdot 2)\sqrt{2} = 6\sqrt{2}$$

c. $7\sqrt{12} = 7\sqrt{4 \cdot 3} = 7(2\sqrt{3}) = 14\sqrt{3}$

d. $\sqrt{c^3} = \sqrt{c^2 \cdot c} = c\sqrt{c}, c \geq 0$

**Try These A**
Simplify each expression.

a. $\sqrt{18}$  

b. $5\sqrt{48}$

c. $\sqrt{126}$

d. $\sqrt{24y^2}$  

e. $\sqrt{45b^3}$
3. Copy the lengths of the crossbars from the chart in Item 1. Then express the lengths of the crossbars in simplified form.

<table>
<thead>
<tr>
<th>Kite</th>
<th>Dimensions of Base (feet)</th>
<th>Exact Length of One Crossbar (feet)</th>
<th>Simplified Form of Length of Crossbar</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 by 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>2 by 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>3 by 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>1 by 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>2 by 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>3 by 6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The process of finding roots can be expanded to cube roots. Finding the cube root of a number or an expression is the inverse operation of cubing that number or expression.

\[
\sqrt[3]{125} = 5, \text{ because } (5)(5)(5) = 125
\]

\[
\sqrt[3]{y^3} = y, \text{ because } (y)(y)(y) = y^3
\]

To simplify cube roots in which the radicand is not a perfect cube, follow the same two-step process that you used for square roots.

**Step 1:** Write the radicand as a product of numbers, one of which is a perfect cube.

**Step 2:** Find the cube root of the perfect cube.

**Example B**

Simplify each expression.

a. \[\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}\]

b. \[3\sqrt[3]{128} = 3\sqrt[3]{64 \cdot 2} = 3\sqrt[3]{64} \cdot \sqrt[3]{2} = 3(4\sqrt[3]{2}) = 12\sqrt[3]{2}\]

c. \[\sqrt[3]{x^5} = \sqrt[3]{x^3 \cdot x^2} = x\sqrt[3]{x^2}\]

**Try These B**

Simplify each expression.

a. \[\sqrt[3]{24}\]

b. \[\sqrt[3]{54z^3}\]

c. \[\sqrt[3]{40b^4}\]
Another way to write radical expressions is with fractional exponents.

6. **Make use of structure.** Use the definition of a radical and the properties of exponents to simplify the expressions of each row of the table. The first row has been done for you.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>Simplified Form</th>
<th>Fractional Exponent Form</th>
<th>Simplified Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{16} \cdot \sqrt{16} )</td>
<td>( 4 \cdot 4 = 16 )</td>
<td>( 16^{\frac{1}{2}} \cdot 16^{\frac{1}{2}} )</td>
<td>( 16^{\frac{1}{2} + \frac{1}{2}} = 16^1 = 16 )</td>
</tr>
<tr>
<td>( \sqrt[3]{8} \cdot \sqrt[3]{8} \cdot \sqrt[3]{8} )</td>
<td>( \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} )</td>
<td>( 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} )</td>
<td>( 81^{\frac{1}{3}} \cdot 81^{\frac{1}{3}} \cdot 81^{\frac{1}{3}} )</td>
</tr>
<tr>
<td>( \sqrt[4]{81} \cdot \sqrt[4]{81} \cdot \sqrt[4]{81} \cdot \sqrt[4]{81} )</td>
<td>( \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} )</td>
<td>( a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} )</td>
<td>( a^{\frac{1}{2} + \frac{1}{2}} = a^1 = a )</td>
</tr>
<tr>
<td>( \sqrt{a} \cdot \sqrt{a} )</td>
<td>( \frac{1}{2} \cdot \frac{1}{2} )</td>
<td>( a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} )</td>
<td>( a^{\frac{1}{2} + \frac{1}{2}} = a^1 = a )</td>
</tr>
</tbody>
</table>

7. Identify and describe any patterns in the table. Write \( a^{\frac{1}{n}} \) as a radical expression.

The general rule for fractional exponents when the numerator is not 1 is \( a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \).

**Example C**

Write \( 6^{\frac{2}{3}} \) as a radical expression.

**Method 1:** \( 6^{\frac{2}{3}} = 6^{2 \cdot \frac{1}{3}} = (6^2)^{\frac{1}{3}} = \sqrt[3]{36} \)

**Method 2:** \( 6^{\frac{2}{3}} = 6^{\frac{1}{3} \cdot 2} = \left(\frac{1}{3} \cdot \frac{2}{3}\right) = \left(\sqrt[3]{6}\right)^2 \)

**Try These C**

Write each of the following as a radical expression.

a. \( 13^{\frac{1}{4}} \)  

b. \( 7^{\frac{3}{2}} \)  

c. \( x^{\frac{3}{2}} \)
Lesson 20-1
Radical Expressions

Check Your Understanding

8. a. What is the value of $16^{\frac{1}{4}}$?
b. What is the value of $16^{\frac{3}{4}}$?

9. For each radical expression, write an equivalent expression with a fractional exponent.
a. $\sqrt[3]{15}$
b. $\sqrt[3]{21}$

LESSON 20-1 PRACTICE

10. A square has an area of 72 square inches. What is its side length $s$? Give the exact answer using simplified radicals.

11. A cube has a volume of 216 cubic centimeters. What is its edge length $s$? Give the exact answer using simplified radicals.

12. A square has an area of $12x^2$ square feet. What is the length of its sides?

13. A cube has a volume of $128y^3$ cubic millimeters. What is its edge length?

14. A kite has a square base with dimensions of 4 feet by 4 feet. What is the length of one of the diagonal crossbars that will be needed to stabilize the kite?

15. For each radical expression, write an equivalent expression with a fractional exponent.
a. $\sqrt[6]{6}$
b. $\sqrt[10]{10}$
c. $\sqrt[5]{5}$
d. $\sqrt[18]{18}$

16. Reason abstractly. Devise a plan for simplifying the fourth root of a number that is not a perfect fourth power. Explain to a friend how to use your plan to simplify the fourth root. Be sure to include examples.
Learning Targets:
- Add radical expressions.
- Subtract radical expressions.

SUGGESTED LEARNING STRATEGIES: Discussion Groups, Close Reading, Note Taking, Think-Pair-Share, Identify a Subtask

The Windy Hill Science Club advisor wants to find the total length of the balsa wood needed to make the frames for the kites. To do so, he will need to add radicals.

Addition Property of Radicals

\[ a\sqrt{b} \pm c\sqrt{b} = (a \pm c)\sqrt{b}, \text{ where } b \geq 0. \]

To add or subtract radicals, the index and radicand must be the same.

Example A

Add or subtract each expression and simplify. State whether the sum or difference is rational or irrational.

a. \[ 3\sqrt{5} + 7\sqrt{5} \]
   \[ = (3 + 7)\sqrt{5} \]
   \[ = 10\sqrt{5} \]
   \[ \text{irrational} \]

b. \[ 10\sqrt{3} - 4\sqrt{3} \]
   \[ = (10 - 4)\sqrt{3} \]
   \[ = 6\sqrt{3} \]
   \[ \text{irrational} \]

c. \[ 2\sqrt{5} + 8\sqrt{3} + 6\sqrt{5} - 3\sqrt{3} \]
   Step 1: Group terms with like radicands. \[ 2\sqrt{5} + 6\sqrt{5} + 8\sqrt{3} - 3\sqrt{3} \]
   Step 2: Add or subtract the coefficients. \[ = (2 + 6)\sqrt{5} + (8 - 3)\sqrt{3} \]
   \[ = 8\sqrt{5} + 5\sqrt{3} \]
   Solution: \[ 2\sqrt{5} + 8\sqrt{3} + 6\sqrt{5} - 3\sqrt{3} = 8\sqrt{5} + 5\sqrt{3} \]; irrational

Try These A

Add or subtract each expression and simplify. State whether the sum or difference is rational or irrational.

a. \[ 2\sqrt{7} + 3\sqrt{7} + \frac{2}{3} \]

b. \[ 5\sqrt{6} + 2\sqrt{5} - \sqrt{6} + 7\sqrt{5} \]

c. \[ 2 + 2\sqrt{2} + \sqrt{8} + 3\sqrt{2} \]
1. The club advisor also needs to know how much wood to buy for the legs of the kites. Each kite will be 3 feet tall.
   a. Complete the table below:

<table>
<thead>
<tr>
<th>Kite</th>
<th>Dimensions of Base (feet)</th>
<th>Length of One Crossbar (feet)</th>
<th>Length of One Leg (feet)</th>
<th>Wood Needed for Legs (feet)</th>
<th>Wood Needed for Crossbars (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 by 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>2 by 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>3 by 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>1 by 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>2 by 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>3 by 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b. **Reason quantitatively.** How much balsa wood should the club advisor buy if the club is going to build the six kites described above? Is the result rational or irrational?

   c. Explain how you reached your conclusion.

2. **Use appropriate tools strategically.** Approximately how much balsa wood, in decimal notation, will the club advisor need to buy?
   a. Use your calculator to approximate the amount of balsa wood, and then decide on a reasonable way to round.

   b. Explain why the club advisor would need this approximation rather than the exact answer expressed as a radical.
Check Your Understanding

Perform the indicated operations. Be sure to completely simplify your answer. State whether each sum or difference is rational or irrational.

3. a. $7\sqrt{6} + 9\sqrt{6}$  
   b. $12\sqrt{4} - 5\sqrt{4} + 2\sqrt{4}$
4. a. $8\sqrt{3} - \sqrt{12} + 3\sqrt{3}$  
   b. $\sqrt{18} + \sqrt{8} + \sqrt{32}$
5. a. $\frac{3}{\sqrt{16}} - \frac{3}{\sqrt{2}}$  
   b. $\frac{3}{\sqrt{3}} + \frac{3}{\sqrt{24}} + 16$
6. a. $3\sqrt{9} + 5\sqrt{9}$  
   b. $3\sqrt{10} + 5\sqrt{10}$
7. Is the sum of a rational number and an irrational number rational or irrational? Support your response with an example.

LESSON 20-2 PRACTICE

Use the figures of a rectangle and kite for Items 8–12.

8. Determine the perimeter of the rectangle.
9. Determine the perimeter of the kite.
10. How much longer is the long side of the rectangle than the longer side of the kite?
11. How much greater is the perimeter of the rectangle than the perimeter of the kite?
12. Make sense of problems. How much wood would be required to insert diagonal crossbars in the rectangle?
Lesson 20-3
Multiplying and Dividing Radical Expressions

Learning Targets:
• Multiply and divide radical expressions.
• Rationalize the denominator of a radical expression.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Predict and Confirm, Discussion Groups, Close Reading, Marking the Text, Note Taking

1. a. Complete the table below and simplify the radical expressions in the third and fifth columns.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>( \sqrt{a} \cdot \sqrt{b} )</th>
<th>ab</th>
<th>( \sqrt{ab} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9</td>
<td>( \sqrt{36} )</td>
<td>36</td>
<td>( \sqrt{36} )</td>
</tr>
<tr>
<td>100</td>
<td>25</td>
<td>( \sqrt{2500} )</td>
<td>2500</td>
<td>( \sqrt{2500} )</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
<td>( \sqrt{144} )</td>
<td>144</td>
<td>( \sqrt{144} )</td>
</tr>
</tbody>
</table>

b. Express regularity in repeated reasoning. Use the patterns you observe in the table above to write an equation that relates \( \sqrt{a} \), \( \sqrt{b} \), and \( \sqrt{ab} \).

c. All the values of \( a \) and \( b \) in part a are perfect squares. In the table below, choose some values for \( a \) and \( b \) that are not perfect squares and use a calculator to show that the equation you wrote in Part (b) is true for those numbers as well.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>( \sqrt{a} \cdot \sqrt{b} )</th>
<th>ab</th>
<th>( \sqrt{ab} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. d. Simplify the products in Columns A and B below.

<table>
<thead>
<tr>
<th>A</th>
<th>Simplified Form</th>
<th>B</th>
<th>Simplified Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (2\sqrt{4}) \sqrt{9} )</td>
<td>( 2\sqrt{36} )</td>
<td>( 2 \cdot 4 )</td>
<td>( 9 )</td>
</tr>
<tr>
<td>( (3\sqrt{4}) \sqrt{16} )</td>
<td>( 3 \cdot 4 )</td>
<td>( 16 )</td>
<td>( \sqrt{576} )</td>
</tr>
<tr>
<td>( (2\sqrt{7}) \sqrt{14} )</td>
<td>( 2 \cdot 7 )</td>
<td>( 14 )</td>
<td>( \sqrt{196} )</td>
</tr>
</tbody>
</table>

e. Which products in the table in Part (d) are rational and which are irrational?
f. Write a verbal rule that explains how to multiply radical expressions.

### Multiplication Property of Radicals

\[
(a\sqrt{b})(c\sqrt{d}) = ac\sqrt{bd},
\]

where \(b \geq 0\), \(d \geq 0\).

To multiply radical expressions, the index must be the same. Find the product of the coefficients and the product of the radicands. Simplify the radical expression.

### Example A

Multiply each expression and simplify.

a. \((3\sqrt{6})(4\sqrt{5}) = (3 \cdot 4)(\sqrt{6 \cdot 5}) = 12\sqrt{30}\)

b. \((2\sqrt{10})(3\sqrt{6})\)

\[
= (2 \cdot 3)\sqrt{10 \cdot 6}
= 6\sqrt{60}
= 6\sqrt{4 \cdot 15}
= 6(\sqrt{4} \cdot \sqrt{15})
= 6(2\sqrt{15})
= 12\sqrt{15}
\]

Step 1: Multiply.

Step 2: Simplify.

To divide radical expressions, the index must be the same. Find the quotient of the coefficients and the quotient of the radicands. Simplify the expression.

### Division Property of Radicals

\[
\frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \sqrt{\frac{b}{d}}
\]

where \(b \geq 0\), \(d \geq 0\).
Lesson 20-3
Multiplying and Dividing Radical Expressions

**Example B**
Divide each expression and simplify.

a. \( \frac{\sqrt{6}}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \)

b. \( \frac{2\sqrt{10}}{3\sqrt{2}} = \frac{2}{3} \cdot \frac{\sqrt{10}}{\sqrt{2}} = \frac{2}{3} \cdot \sqrt{5} \)

c. \( \frac{8\sqrt{24x^2}}{2\sqrt{3}} = \frac{8}{2} \cdot \frac{\sqrt{24x^2}}{\sqrt{3}} = 4\sqrt{8x^2} = 4\cdot \sqrt{4} \cdot \sqrt{2} \cdot x^2 = 4(2x\sqrt{2}) = 8x\sqrt{2} \)

**Try These B**
Divide each expression and simplify.

a. \( \frac{4\sqrt{42}}{5\sqrt{6}} \)

b. \( \frac{10\sqrt{54}}{2\sqrt{2}} \)

c. \( \frac{12\sqrt{75}}{3\sqrt{3}} \)

d. \( \frac{16\sqrt{3x^{11}}}{8\sqrt{x^2}} \)

A radical expression in simplified form does not have a radical in the denominator. Most frequently, the denominator is rationalized. You **rationalize the denominator** by simplifying the expression to get a perfect square under the radicand in the denominator.

\[
\frac{\sqrt{a}}{\sqrt{b}} \cdot 1 = \left( \frac{\sqrt{a}}{\sqrt{b}} \right) \left( \frac{\sqrt{b}}{\sqrt{b}} \right) = \frac{\sqrt{ab}}{\sqrt{b^2}} = \frac{\sqrt{ab}}{b}
\]

**Example C**
Rationalize the denominator of \( \frac{\sqrt{5}}{\sqrt{3}} \).

Step 1: Multiply the numerator and denominator by \( \sqrt{5} \).

\( \frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{15}}{\sqrt{9}} = \frac{\sqrt{15}}{3} \)

Step 2: Simplify.

Solution: \( \frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{15}}{3} \)

**Try These C**
Rationalize the denominator in each expression.

a. \( \frac{\sqrt{11}}{\sqrt{6}} \)

b. \( \frac{2\sqrt{7}}{\sqrt{5}} \)

c. \( \frac{3\sqrt{5}}{\sqrt{8}} \)
Check Your Understanding

Express each expression in simplest radical form. State whether each result in Items 2–5 is rational or irrational.

2. \((4\sqrt{7})(2\sqrt{3})\)

3. \(\sqrt{2}(\sqrt{2} + 3\sqrt{6})\)

4. \(\frac{\sqrt{75}}{\sqrt{5}}\)

5. \(\sqrt{\frac{5}{8}}\)

6. \((3\sqrt{32y})(4\sqrt{25y})\)

7. \(\frac{6\sqrt{98x^4}}{\sqrt{2x}}\)

8. Is the product of a nonzero rational number and an irrational number rational or irrational? Support your response with an example.

LESSON 20-3 PRACTICE

Express each expression in simplest radical form. State whether each result in Items 9–12 is rational or irrational.

9. \(\left(\frac{1}{\sqrt{2}}\right)\left(\frac{3}{\sqrt{5}}\right)\)

10. \(\sqrt{27} \cdot \frac{1}{\sqrt{27}}\)

11. \(\frac{2\sqrt{7}}{\sqrt{3}}\)

12. \(\frac{4\sqrt{5}}{3\sqrt{2}}\)

13. \(\left(4\sqrt[3]{4m^2}\right)\left(5m\sqrt[3]{2m^2}\right)\)

14. \(\frac{2\sqrt{52x^9}}{\sqrt{13x}}\)

15. **Attend to precision.** What conditions must be satisfied for a radical expression to be in simplified form?
ACTIVITY 20 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 20-1

Write each expression in simplest radical form.

1. \( \sqrt{40} \)
2. \( \sqrt{128} \)
3. \( \sqrt{162} \)

Use the Pythagorean Theorem and the triangle below for Items 4 and 5. Recall that the Pythagorean Theorem states that for all right triangles, \( a^2 + b^2 = c^2 \).

4. In the right triangle, if \( a = 3 \) and \( b = 6 \), what is the value of \( c \)?
   A. \( 3\sqrt{5} \)  
   B. 9  
   C. \( 9\sqrt{5} \)  
   D. 45

5. In the right triangle, if \( a = 12 \) and \( b = 15 \), what is the value of \( c \)?

Simplify each expression.

6. \( \sqrt{4m^7} \)
7. \( 3\sqrt[3]{16n^8} \)
8. \( \frac{3}{2}\sqrt[3]{16x^4} \)

Write each of the following as a radical expression.

9. \( 15\frac{2}{3} \)
10. \( (2p)^{\frac{1}{3}} \)
11. \( 16x^{\frac{3}{4}} \)

Lesson 20-2

Write each expression in simplest radical form. State whether each result is rational or irrational.

12. Which of the following expressions is not equivalent to \( (8x)^{\frac{2}{3}} \)?
   A. \( 4\sqrt[3]{x^2} \)  
   B. \( \frac{3}{2}\sqrt[3]{x^2} \)  
   C. \( 4x^{\frac{2}{3}} \)  
   D. \( \frac{2}{3}\sqrt[3]{x^3} \)

13. \( 4\sqrt{27} + 6\sqrt{12} \)
14. \( 8\sqrt{6} + 2\sqrt{12} + 5\sqrt{3} - \sqrt{54} \)
15. \( 3\sqrt{36} - 5\sqrt{16} + 4 \)
16. Which of the following is the difference of \( 9\sqrt{20} \) and \( 2\sqrt{5} \)?
   A. \( 7\sqrt{15} \)  
   B. \( 16\sqrt{5} \)  
   C. \( 9\sqrt{5} \)  
   D. \( 7\sqrt{5} \)

The figure below is composed of a rectangle and a right triangle. Use the figure for Items 17–19.

17. Determine the perimeter of the rectangle.
18. Determine the perimeter of the triangle.
19. Determine the perimeter of the composite figure.
20. A student was asked to completely simplify the expression \( 3\sqrt{3} + \sqrt{12} + 2\sqrt{3} \). The student wrote \( 5\sqrt{3} + \sqrt{12} \). Do you agree with the student’s answer? Explain.
Lesson 20-3

The figure shows a rectangular prism. The volume of the rectangular prism is the product of the length, width, and height. Use the figure for Items 21–24.

21. If \( l = \sqrt{3} \), \( w = \sqrt{2} \), and \( h = \sqrt{6} \), what is the volume of the rectangular prism? Is the volume rational or irrational?

22. If \( l = 3\sqrt{3} \), \( w = 2\sqrt{2} \), and \( h = 5\sqrt{10} \), what is the volume of the rectangular prism? Is the volume rational or irrational?

23. If the volume of the rectangular prism is 20, the length is \( \sqrt{3} \), and the width is \( \sqrt{5} \), what is the height?

24. If the volume of the rectangular prism is \( 24\sqrt{3} \), the height is \( 2\sqrt{2} \), and the width is \( 3\sqrt{10} \), what is the length?

Write each expression in simplest form.

25. \( \left( 2\sqrt{2x^2} \right) \left( 3x\sqrt{x^2} \right) \)

26. \( \left( 6p\sqrt{p^3} \right) \left( 0.2\sqrt{16p} \right) \)

27. \( \left( 4\sqrt{8m} \right) \left( 7m\sqrt{m^2} \right) \)

28. \( \frac{3\sqrt{32}}{\sqrt{2}} \)

29. \( \frac{\sqrt{81x}}{3} \)

30. \( \frac{\sqrt{x^2y^5}}{\sqrt{y}} \)

31. Which of the following expressions cannot be simplified any further?

A. \( \frac{5}{2} \)  
B. \( \frac{5}{\sqrt{2}} \)
C. \( \sqrt{52} \)  
D. \( 5\sqrt{2} \)

32. Elena was asked to simplify the expression \( \left( 2\sqrt{12x} \right) \left( 4\sqrt{3} \right) \). Her answer was \( 48x\sqrt{3} \).

a. Explain how Elena can use her calculator to check whether her answer is reasonable.

b. Is Elena’s answer correct? If not, explain Elena’s mistake and give the correct answer.

The time, \( T \), in seconds, it takes the pendulum of a clock to swing from one side to the other side is given by the formula \( T = \frac{\pi l}{\sqrt{32}} \), where \( l \) is the length of the pendulum, in feet. The clock ticks each time the pendulum is at the extreme left or right point.

Use this information for Items 33–36.

33. If the pendulum is 4 feet long, how long does it take the pendulum to swing from left to right? Give an exact value in terms of \( \pi \).

34. If the pendulum is 8 feet long, how long does it take the pendulum to swing from left to right? Give an exact value in terms of \( \pi \).

35. If the pendulum is shortened, will the clock tick more or less often? Explain how you arrived at your conclusion.

36. Approximately what length of the pendulum will result in its swinging from one side to the other every second?

MATHEMATICAL PRACTICES

Construct Viable Arguments and Critique the Reasoning of Others

37. Amil knows that the formula for the area of a circle is \( A = \pi r^2 \). He says that the area of a circle with a radius of 2\( \sqrt{5} \) feet is \( 4\sqrt{5}\pi \) square feet. Is he correct? If not, describe his error.
For her Electronic Communications class, Keisha has been tasked with investigating the effects of social media. She decides to post a video in cyberspace to see if she can make it go viral.

To get things started, Keisha e-mails the video link to three of her friends. In the message, she asks each of the recipients to forward the link to three of his or her friends. Whenever a recipient forwards the link, Keisha asks him or her to attach the following message: *After watching, please forward this video link to three of your friends who have not yet received it.*

One way to visually represent this situation is with a tree diagram. A *tree diagram* shows all the possible outcomes of an event.

1. Use the tree diagram to help you complete the table. (Assume that everyone who receives the video link watches the video.)

<table>
<thead>
<tr>
<th>Stage</th>
<th>Number of People Who Watch the Video</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

2. *Express regularity in repeated reasoning.* Describe any patterns you notice in the table.
3. Use the table of values to graph the viral video situation.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Number of People Who Watch Video</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
</tr>
</tbody>
</table>


5. Is the graph the graph of a function? If so, what is the domain?

The number of people who have received the video link at each stage form a geometric sequence. A geometric sequence is a sequence of values in which a nonzero constant ratio exists between consecutive terms. The constant ratio is called the common ratio and is typically denoted by the letter $r$. The common ratio is the value that each term is multiplied by to get the next term.

6. a. Write the numbers of people who have received the video as a sequence.

Lesson 21-1
Identifying Geometric Sequences

7. Identify each sequence as arithmetic, geometric, or neither. If it is arithmetic, state the common difference. If it is geometric, state the common ratio.
   a. 5, 8, 11, 14, …
   b. 18, 6, 2, \( \frac{2}{3} \), …
   c. 1, 4, 9, 16, …
   d. \(-1, 4, -16, 64, \ldots\)
   e. 16, \(-8, 4, -2, \ldots\)

Check Your Understanding

Identify each sequence as arithmetic, geometric, or neither. If it is arithmetic, state the common difference. If it is geometric, state the common ratio.

8. 10, 8, 6, 4, …
9. 2, \( \frac{1}{2} \), \( \frac{1}{8} \), \( \frac{1}{32} \), …
10. 9, \(-3, 1 - \frac{1}{3}, \ldots\)

LESSON 21-1 PRACTICE

A cell divides in two every day. The tree diagram shows the first few stages of this process. Use the tree diagram for Items 11–13.

Day 1

Day 2

Day 3

11. Make a table of values to represent the scenario shown in the tree diagram.
12. Does the tree diagram represent a geometric sequence? If so, what is the common ratio?
13. If the diagram were extended to a sixth day, how many circles would there be on Day 6?
14. Reason abstractly. Can a geometric sequence ever have a term equal to 0? Explain.
Learning Targets:

• Write a recursive formula for a geometric sequence.
• Write an explicit formula for a geometric sequence.
• Use a formula to find a given term of a geometric sequence.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Discussion Groups, Think-Pair-Share, Construct an Argument, Sharing and Responding

Remember that the numbers in a sequence are called terms, and you can use sequence notation \( a_n \) or function notation \( f(n) \) to refer to the \( n \)th term.

1. a. Use the above notation to rewrite the first four terms of the viral video sequence. Also write the common ratio.

\[
a_1 = f(1) = \quad a_2 = f(2) = \quad a_3 = f(3) = \quad a_4 = f(4) = \quad r = \]

b. What is the value of the term following \( a_4 = f(4) \)? Write an expression to represent this term using \( a_4 \) and the common ratio.

Just as with arithmetic sequences, you can use a recursive formula to represent a geometric sequence.

2. Complete the table below for the viral video sequence.

<table>
<thead>
<tr>
<th>Term</th>
<th>Sequence Representation Using Common Ratio</th>
<th>Function Representation Using Common Ratio</th>
<th>Numerical Value (number of people who have seen the video)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 = f(1) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( a_2 = f(2) )</td>
<td>( 3(1) = 3a_1 )</td>
<td>( 3(1) = 3f(1) )</td>
<td>3</td>
</tr>
<tr>
<td>( a_3 = f(3) )</td>
<td>( 3(3) = 3a_2 )</td>
<td>( 3(3) = 3f(2) )</td>
<td></td>
</tr>
<tr>
<td>( a_4 = f(4) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a_5 = f(5) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( a_n = f(n) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. The recursive formulas for the viral video sequence are partially given below. Complete the formulas by writing the expressions for \( a_n \) and \( f(n) \).

\[
\begin{align*}
a_1 &= 1 \\
\quad a_n &= \quad f(1) &= 1 \\
\quad f(n) &= 
\end{align*}
\]
Lesson 21-2
Formulas for Geometric Sequences

Check Your Understanding

Write a recursive formula for each geometric sequence. Include the recursive formula in function notation.

4. 64, 16, 4, 1, . . .
5. 64, −16, 4, −1, . . .
6. −1, 1, −1, 1, . . .

7. Use the recursive formula to find \(a_6\), \(a_7\), and \(a_8\) for the viral video sequence. Explain your results.

8. Why might it be difficult to find the 100th term of the viral video sequence using the recursive formula?

As with arithmetic sequences, geometric sequences can be represented with explicit formulas. The terms in a geometric sequence can be written as the product of the first term and a power of the common ratio.

9. For the viral video sequence, identify \(a_1\) and \(r\). Then fill in the missing exponents and blanks.

\[
\begin{align*}
    a_1 &= \underline{\quad} \\
    a_2 &= 1 \cdot 3^{\underline{\quad}} = 3 \\
    a_3 &= 1 \cdot 3^{\underline{\quad}} = 9 \\
    a_4 &= 1 \cdot 3^{\underline{\quad}} = \underline{\quad} \\
    a_5 &= 1 \cdot 3^{\underline{\quad}} = \underline{\quad} \\
    a_6 &= 1 \cdot 3^{\underline{\quad}} = \underline{\quad} \\
    a_{10} &= 1 \cdot 3^{\underline{\quad}} = \underline{\quad}
\end{align*}
\]

10. **Express regularity in repeated reasoning.** Describe any patterns you observe in your responses to Item 9. Then use \(a_1\), \(r\), and \(n\) to write a formula for the \(n\)th term of any geometric sequence.

11. Write the explicit formula for the viral video sequence. Use the formula to determine the 12th term of the sequence. What does the 12th term represent?
12. The explicit formula for a geometric sequence can be thought of as a function.
   a. What is the input? What is the output?

   b. State the domain of the function.

   c. Rewrite the explicit formula for the viral video sequence using function notation.

   d. Use appropriate tools strategically. Use a graphing calculator to graph your function from Part (c). Is the function linear or nonlinear? Justify your response.

13. Consider the geometric sequence 5, 10, 20, 40, . . .
   a. Write the explicit formula for the sequence.

   b. How can you check that your formula is correct?

   c. Determine the 16th term in the sequence.

   d. Use function notation to write the explicit formula for the sequence.

   e. What is the value of \( f(10) \)? What does it represent?

14. a. Write the explicit formula for the geometric sequence 32, 16, 8, 4, . . .

   b. Determine the 9th term in the sequence.
Lesson 21-2
Formulas for Geometric Sequences

15. The explicit formula for a geometric sequence is \( a_n = 6 \cdot 3^{n-1} \). State the recursive formula for the sequence. Include the recursive formula in function notation.

Check Your Understanding

16. How can you use the recursive formula for a geometric sequence to write the explicit formula?

Write the explicit formula for each geometric sequence. Then determine the 6th term of each sequence.

17. \( 1, 5, 25, \ldots \)
18. \( 48, -24, 12, \ldots \)
19. \( \begin{align*}
    a_1 &= -81 \\
    a_n &= -\frac{1}{3}a_{n-1}
\end{align*} \)

20. Make sense of problems. Revisit the viral video scenario at the beginning of the activity. How many stages will it take until 1 million new people receive the link to the viral video? Explain how you found your answer.

Check Your Understanding

21. Write a recursive formula for the geometric sequence whose explicit formula is \( a_n = 1 \cdot (-2)^{n-1} \). Include the recursive formula in function notation.

22. Write an explicit formula for the sequence \( \begin{align*}
    a_1 &= 3 \\
    a_n &= 2a_{n-1}
\end{align*} \).
LESSON 21-2 PRACTICE

The diagram below shows a square repeatedly divided in half. The entire square has an area of 1 square unit. The number in each region is the area of the region. Use the diagram for Items 23–25.

23. Write a geometric sequence to describe the areas of successive regions.

24. Write an explicit formula for the geometric sequence that you wrote in Item 23.

25. Model with mathematics. What is the 10th term of the sequence? What does it represent?

26. The explicit formula for a geometric sequence is $f(n) = 5(-2)^{n-1}$. Give the recursive formula for the sequence.
ACTIVITY 21 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 21-1
In Items 1 and 2, assume that the first term of a sequence is \(-3\).

1. Write the first four terms of the sequence if it is an arithmetic sequence with common difference \(-\frac{1}{3}\).

2. Write the first four terms of the sequence if it is a geometric sequence with common ratio \(-\frac{1}{3}\).

The tree diagram below shows the number of possible outcomes when tossing a coin a number of times. For example, if you toss a coin once (Stage 1), there are two possible outcomes: heads (H) and tails (T). If you toss a coin twice (Stage 2), there are four possible outcomes for the two tosses: HH, HT, TH, and TT.

Use the tree diagram for Items 3–5.

3. How many possible outcomes are there when you toss a coin 4 times?

4. Identify the common ratio of the sequence represented by the tree diagram.

5. How many possible outcomes are there when you toss a coin 23 times? Express your answer using exponents.

For Items 6–10, identify each sequence as arithmetic, geometric, or neither. If it is arithmetic, state the common difference. If it is geometric, state the common ratio.

6. 17, 25, 33, 41, . . .

7. 1, 3, 6, 10, 15, . . .

8. \(-27, -9, -3, -1, \ldots\)

9. 0.1, 0.5, 0.9, 1.3, . . .

10. \(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\)

11. A geometric sequence begins with the value 1 and has a common ratio of \(-2\). Identify the eighth term in the sequence.

A. \(-128\)  B. 128
C. \(-256\)  D. 256

12. A geometric sequence begins with the value 2 and has a common ratio of \(\frac{1}{2}\). Identify the fifth term in the sequence.

A. 4  B. \(\frac{1}{4}\)
C. \(\frac{1}{8}\)  D. \(\frac{1}{16}\)

13. Which of the following is a false statement about the sequence 2, 4, 8, 16, 32, . . . ?

A. The common ratio of the sequence is 2.
B. The tenth term of the sequence is \(2^{10}\).
C. Every term of the sequence is even.
D. The number 216 appears in the sequence.

14. Give an example of a geometric sequence with a common ratio of 0.2. Write at least the first four terms of the sequence.
Lesson 21-2
Write a recursive formula for each geometric sequence. Include the recursive formula in sequence notation.

15. 7, 21, 63, 189, . . .
16. 100, 10, 1, 0.1, . . .
17. −10, 20, −40, 80, . . .

Write the explicit formula for each geometric sequence. Then determine the 8th term of each sequence.

18. 4, 16, 64, 256, . . .
19. \( \frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots \)
20. \[
\begin{cases}
 a_1 = 5 \\
 a_n = -3a_{n-1}
\end{cases}
\]

A contestant on a game show wins $100 for answering a question correctly in Round 1. In each subsequent round, the contestant’s winnings are doubled if she gives a correct answer. If the contestant gives an incorrect answer, she loses everything. Use this information for Items 21–23.

21. Write an explicit formula that gives the contestant’s winnings in round \( n \), assuming she answers all questions correctly.

22. How much does the contestant win in Round 10, assuming she answers all questions correctly?

23. How many rounds does a contestant need to play in order to answer a question worth at least $1,000,000?

24. A geometric sequence is given by the recursive formula \[
\begin{cases}
 f(1) = -6 \\
 f(n) = -\frac{1}{2} f(n - 1)
\end{cases}
\] Which of the following is a term in the sequence?

A. \( \frac{3}{4} \)  
B. \( -\frac{3}{4} \)  
C. \( \frac{3}{2} \)  
D. \( -3 \)

Each time a pendulum swings, the distance it travels decreases, as shown in the figure.

The table shows how far the pendulum travels with each swing. Use this table for Items 25–27.

<table>
<thead>
<tr>
<th>Swing Number</th>
<th>Distance Traveled (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
</tr>
<tr>
<td>4</td>
<td>33.75</td>
</tr>
</tbody>
</table>

25. Write the explicit formula for the pendulum situation.

26. How far will the pendulum travel on the seventh swing?

27. How many swings will it take for the pendulum to travel less than 10 cm?

The game commission observes the fish population in a stream and notices that the number of trout increases by a factor of 1.5 every week. The commission initially observed 80 trout in the stream. Use this information for Items 28–31.

28. Write the explicit formula for the trout situation.

29. Make a graph of the population growth.

30. If this pattern continues, how many trout will be in the stream on the fifth week?

31. If this pattern continues, on what week will the trout population exceed 500?

MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

32. Samir says that it is possible for a sequence to be both an arithmetic sequence and a geometric sequence. Do you agree or disagree? Explain.
Stocking a lake is the process of adding fish to the lake. Once the fish have been added to the lake, their population growth depends upon many factors, such as the species of the fish, the number of predators in the lake, the quality of the water, and the lake’s food supply.

Sapphire Lake was stocked with five species of fish several years ago. Michelle, a new employee at the parks and recreation commission, wants to analyze the population growth of the fish. She is able to find only the information shown below.

<table>
<thead>
<tr>
<th>Species</th>
<th>Population Model</th>
<th>Meaning of Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$(3x^2y)^2$</td>
<td>$x =$ average length of species in centimeters $y =$ year of project (Year 1 is the year in which fish are first added to the lake.)</td>
</tr>
<tr>
<td>B</td>
<td>$6(xy)^3$</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>$20\sqrt{xy}$</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>$4\sqrt{x^2y}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Species E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

1. Michelle wants to know the ratio of the population of species A to species B.
   a. Write the ratio as a fraction in simplified form without negative exponents.
   b. Write the ratio using negative exponents.

2. Next, Michelle analyzes the population of species C and D.
   a. Write the ratio of the population of species C to species D as a fraction in simplified form.
   b. Michelle needs to know the total population of species C and D in Year 1. She learns that the average length of both species is 8 centimeters. Write a simplified expression for the approximate total population of species C and D in Year 1.
   c. Is the expression you wrote in Part (b) rational or irrational? Explain your reasoning.

3. Michelle assumes that the population of species E continues to grow as shown in the table.
   a. Write an explicit formula for the sequence.
   b. Write a recursive formula for the sequence. Include the recursive formula in function notation.
   c. According to the model, what was the approximate population of species E in year 7?
### Scoring Guide

<table>
<thead>
<tr>
<th>Mathematics Knowledge and Thinking (Items 1a, 1b, 2a, 2b, 3c)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clear and accurate understanding of how to write and simplify exponential and radical expressions, including expressions with negative exponents</td>
<td>Adequate understanding of how to write and simplify exponential and radical expressions, including expressions with negative exponents</td>
<td>Partial understanding of how to write and simplify exponential and radical expressions, including expressions with negative exponents</td>
<td>Little or no understanding of how to write and simplify exponential and radical expressions, including expressions with negative exponents</td>
<td></td>
</tr>
<tr>
<td>Fluency in determining a specified term of a geometric sequence</td>
<td>Correct identification of a specified term of a geometric sequence</td>
<td>Partial understanding of and some difficulty determining a specified term of a geometric sequence</td>
<td>Incomplete understanding of and significant difficulty determining a specified term of a geometric sequence</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem Solving (Item 3)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Appropriate and efficient strategy that results in a correct answer</td>
<td>Strategy that may include unnecessary steps but results in a correct answer</td>
<td>Strategy that results in some incorrect answers</td>
<td>No clear strategy when solving problems</td>
</tr>
</tbody>
</table>

| Mathematical Modeling/ Representations (Items 3a, 3b) | | |
|-------------------------------------------------------|-----------|
| Effective understanding of how to describe a real-world data set using explicit and recursive formulas | Little difficulty describing a real-world data set using explicit and recursive formulas | Partial understanding of how to describe a real-world data set using explicit and recursive formulas | Inaccurate or incomplete understanding of how to describe a real-world data set using explicit and recursive formulas |

<table>
<thead>
<tr>
<th>Reasoning and Communication (Items 2c, 3c)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Precise use of appropriate math terms and language to explain whether an expression is rational or irrational</td>
<td>Correct identification of an expression as rational or irrational with an adequate explanation</td>
<td>Misleading or confusing explanation of whether an expression is rational or irrational</td>
<td>Incomplete or inaccurate explanation of whether an expression is rational or irrational</td>
</tr>
<tr>
<td>Ease and accuracy describing the relationship between a sequence and a real-world scenario</td>
<td>Little difficulty describing the relationship between a sequence and a real-world scenario</td>
<td>Partially correct description of the relationship between a sequence and a real-world scenario</td>
<td>Little or no understanding of how a sequence might relate to a real-world scenario</td>
</tr>
</tbody>
</table>
Learning Targets:
• Understand the definition of an exponential function.
• Graph and analyze exponential growth functions.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Look for a Pattern, Interactive Word Wall, Predict and Confirm, Think-Pair-Share

The National Association of Realtors estimates that, on average, the price of a house doubles every ten years. Tony’s grandparents bought a house in 1960 for $10,000. Assume that the trend identified by the National Association of Realtors applies to Tony’s grandparents’ house.

1. What was the value of Tony’s grandparents’ house in 1970 and in 1980?

2. Compute the difference in value from 1960 to 1970.

3. Compute the ratio of the 1970 value to the 1960 value.

4. Complete the table of values for the years 1960 to 2010.

<table>
<thead>
<tr>
<th>Year</th>
<th>Decades Since 1960</th>
<th>Value of House</th>
<th>Difference Between Values of Consecutive Decades</th>
<th>Ratio of Values of Consecutive Decades</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>0</td>
<td>$10,000</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>1970</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1980</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1990</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2010</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. What patterns do you recognize in the table?
6. Write the house values as a sequence. Identify the sequence as arithmetic or geometric and justify your answer.

7. Using the data from the table, graph the ordered pairs (decades since 1960, house value) on the coordinate grid below.

8. The data comparing the number of decades since 1960 and value of the house are not linear. Explain why using the table and the graph.

9. Make use of structure. Using the information that you have regarding the house value, predict the value of the house in the year 2020. Explain how you made your prediction.

10. Tony would like to know what the value of the house was in 2005. Using the same data, predict the house value in 2005. Explain how you made your prediction.

The increase in house value for Tony’s grandparents’ house is an example of **exponential growth**. Exponential growth can be modeled using an **exponential function**.

<table>
<thead>
<tr>
<th>Exponential Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>A function of the form $f(x) = a \cdot b^x$, where $x$ is the domain value, $f(x)$ is the range value, $a \neq 0$, $b &gt; 0$, and $b \neq 1$.</td>
</tr>
</tbody>
</table>
Lesson 22-1
Exponential Functions and Exponential Growth

In exponential growth, a quantity is multiplied by a constant factor greater than 1 during each time period.

11. The value of Tony’s grandparents’ house is growing exponentially because it is multiplied by a constant factor for each decade. What is this constant factor?

A function that can be used to model the house value is \( h(t) = 10,000 \cdot (2)^t \). Use this function for Items 12–17.

12. Identify the meaning of \( h(t) \) and \( t \). What are the reasonable domain and range?

13. Describe how your answer to Item 11 is related to the function \( h(t) = 10,000 \cdot (2)^t \).

14. Complete the table of values for \( t \) and \( h(t) \). Then graph the function \( h(t) \) on the grid below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( h(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

15. What was the value of the house in 1960? Describe how this value is related to the function \( h(t) = 10,000 \cdot (2)^t \) and to the graph.

16. Calculate the value of the house in the year 2020. How does the value compare with your prediction in Item 9?

17. Calculate the value of the house in the year 2005. How does the value compare with your prediction in Item 10?
The value of houses in different locations can grow at different rates. The table below shows the value of Maddie’s house from 1960 until 2010. Use the table for Items 23–25.

<table>
<thead>
<tr>
<th>Year</th>
<th>Decades Since 1960</th>
<th>Value of House</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>0</td>
<td>$10,000</td>
</tr>
<tr>
<td>1970</td>
<td>1</td>
<td>$15,000</td>
</tr>
<tr>
<td>1980</td>
<td>2</td>
<td>$22,500</td>
</tr>
<tr>
<td>1990</td>
<td>3</td>
<td>$33,750</td>
</tr>
<tr>
<td>2000</td>
<td>4</td>
<td>$50,625</td>
</tr>
<tr>
<td>2010</td>
<td>5</td>
<td>$75,938</td>
</tr>
</tbody>
</table>

23. Create a graph showing the value of Maddie’s house from 1960 until 2010.

24. Explain how you know that the value of Maddie’s house is growing exponentially.

25. What was the approximate value of Maddie’s house in 1995?

The function $f(t) = 20,000 \cdot (1.2)^t$ can be used to find the value of Eduardo’s house between 1970 and 2010, where the initial value of the function is the value of Eduardo’s house in 1970.

26. **Model with mathematics.** Describe what the domain and range of the function mean in the context of Eduardo’s house value.

27. What was the value of Eduardo’s house in 1970?

28. Approximately how much was the house worth in 2000?
Radon, a naturally occurring radioactive gas, was identified as a health hazard in some homes in the mid 1980s. Since radon is colorless and odorless, it is important to be aware of the concentration of the gas. Radon has a *half-life* of approximately four days.

Tony’s grandparents’ house was discovered to have a radon concentration of 400 pCi/L. Renee, a chemist, isolated and eliminated the source of the gas. She then wanted to know the quantity of radon in the house in the days following so that she could determine when the house would be safe.

1. **Make sense of problems.** What is the amount of the radon in the house four days after the source was eliminated? Explain your reasoning.

2. Compute the difference in the amount of radon from Day 0 to Day 4.

3. Determine the ratio of the amount of radon on Day 4 to the amount of radon on Day 0.
4. Complete the table for the radon concentration.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>400</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Express regularity in repeated reasoning. What patterns do you recognize in the table?

6. Graph the data in the table as ordered pairs in the form (half-lives, concentration).

7. The data that compares the number of half-lives and the concentration of radon are not linear. Explain why using the table of values and the graph.
Lesson 22-2
Exponential Decay

8. Renee needs to know the concentration of radon in the house after 20 days. How many radon half-lives are in 20 days? What is the concentration after 20 days?

9. How many radon half-lives are in 22 days? Predict the concentration after 22 days.

The decrease in radon concentration in Tony’s grandparents’ house is an example of exponential decay. Exponential decay can be modeled using an exponential function.

In exponential decay, a quantity is multiplied by a constant factor that is greater than 0 but less than 1 during each time period.

10. a. The concentration of radon is multiplied by a constant factor for each half-life. What is this constant factor?

b. Write an exponential function \( r(t) \) for the table in Item 4, using 400 as the initial concentration of radon and the constant factor from part a.

Use the function from Item 10b for Items 11–17.

11. Identify the meaning of \( r(t) \) and \( t \). What are the reasonable domain and range?

12. Describe how your answer to Item 10a is related to the function \( r(t) \).

13. Graph the function \( r(t) \).

14. Describe how the original concentration of radon is related to the function and to the graph.
Lesson 22-2
Exponential Decay

My Notes

15. Use the function to identify the concentration of radon after 20 days. How does the concentration compare with your prediction in Item 8?

16. Use the function to calculate the concentration of radon after 22 days. How does the concentration compare with your prediction in Item 9?

17. Construct viable arguments. Will the concentration of radon ever be 0? Explain your reasoning.

Check Your Understanding

18. Copy and complete the table for the exponential function \( g(x) = \left(\frac{1}{4}\right)^x \).

19. Identify the constant factor for this exponential function.

### LESSON 22-2 PRACTICE

20. The amount of medication in a patient’s bloodstream decreases exponentially from the time the medication is administered. For a particular medication, a function that gives the amount of medication in a patient’s bloodstream \( t \) hours after taking a 100 mg dose is \( A(t) = 100 \left(\frac{7}{10}\right)^t \). Use this function to find the amount of medication remaining after 2 hours.

21. Make a table of values and graph each function.
   a. \( h(x) = 2^x \)
   b. \( l(x) = 3^x \)
   c. \( m(x) = \left(\frac{1}{2}\right)^x \)
   d. \( p(x) = \left(\frac{1}{3}\right)^x \)

22. Which of the functions in Item 21 represent exponential growth? Which of the functions represent exponential decay? Explain using your table of values and graph.

23. Reason abstractly. How can you identify which of the functions represent growth or decay by looking at the function?

24. Express regularity in repeated reasoning. Write an exponential function and identify its constant factor.
Learning Targets:
- Describe key features of graphs of exponential functions.
- Compare graphs of exponential and linear functions.

SUGGESTED LEARNING STRATEGIES: Predict and Confirm, Discussion Groups, Create Representations, Look for a Pattern, Sharing and Responding, Summarizing

Recall that an exponential function is a function of the form \( f(x) = ab^x \), where \( a \neq 0 \), \( b > 0 \), and \( b \neq 1 \).

1. Use a graphing calculator to graph each function. Sketch each graph on the coordinate grid provided.

<table>
<thead>
<tr>
<th>( a &gt; 0 )</th>
<th>( a &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( y = 3 \cdot 2^x )</td>
<td>b. ( y = (-3) \cdot 2^x )</td>
</tr>
<tr>
<td>( b &gt; 1 )</td>
<td>( b &gt; 1 )</td>
</tr>
<tr>
<td>( 0 &lt; b &lt; 1 )</td>
<td>( 0 &lt; b &lt; 1 )</td>
</tr>
</tbody>
</table>

   c. \( y = 3 \cdot (0.5)^x \)
   d. \( y = -2 \cdot (0.5)^x \)

2. Compare and contrast the graphs and equations in Items 1a and 1b above.
   a. How are the equations similar and different?
b. Use words like *increasing*, *decreasing*, *positive*, *negative*, *domain*, and *range* to describe the similarities and differences in the graphs.

c. What connections can be made between the graphs and their equations?

3. Compare and contrast the graphs and equations in Items 1c and 1d.
   a. How are the equations similar and different?
   b. Use words like *increasing*, *decreasing*, *positive*, *negative*, *domain*, and *range* to describe the similarities and differences between the graphs.
   c. What connections can be made between the graphs and their equations?

4. Describe the effects of the values of $a$ and $b$ on the graph of the exponential function $f(x) = ab^x$.
   a. Describe the graph of an exponential function when $a > 0$.
   b. Describe the graph of an exponential function when $a < 0$.
   c. Describe the graph of an exponential function when $b > 1$.
   d. Describe the graph of an exponential function when $0 < b < 1$. 
Lesson 22-3
Graphs of Exponential Functions

Check Your Understanding

5. Describe the values of \( a \) and \( b \) for which the exponential function \( f(x) = ab^x \) is always positive.

6. Describe the values of \( a \) and \( b \) for which the exponential function \( f(x) = ab^x \) is increasing.

7. Let \( f(x) = 2x \) and \( g(x) = 2^x \). Complete the tables below for each function.

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th></th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. Graph \( f(x) \) and \( g(x) \) below.

9. Examine the graphs of \( f(x) \) and \( g(x) \). Compare the values of each function from \( x = -2 \) through \( x = 2 \). Which function is greater on this interval?

10. Examine the values of \( f(x) \) and \( g(x) \) for \( x > 2 \).
    a. Which function is greater on this interval?
b. Do you think this will continue to be true as \( x \) continues to increase? Explain your reasoning.

11. To take a closer look at the graphs of \( f(x) \) and \( g(x) \) for larger values of \( x \), regraph the two functions below. Note the new scale.

12. Does the new graph support the prediction you made in Item 10b?

13. Which function increases faster, \( f(x) \) or \( g(x) \)? Explain your reasoning using the graph and the tables.

Alex believes that for the linear function \( f(x) = 50x \) and the exponential function \( g(x) = 2^x \), the value of \( f(x) \) is always greater than the value of \( g(x) \).

Glenda believes that for a linear function \( f(x) \) to always be greater than an exponential function \( g(x) \), the graph of \( f(x) \) must be very steep while the graph of \( g(x) \) must be very flat. She proposes graphing \( f(x) = 50x \) and \( g(x) = 1.1^x \) to test her conjecture.

14. a. Test Alex’s conjecture by graphing \( f(x) = 50x \) and \( g(x) = 2^x \) on your graphing calculator. Do you agree or disagree with Alex’s conjecture? Explain your reasoning.
Lesson 22-3
Graphs of Exponential Functions

b. Use appropriate tools strategically. Now adjust your viewing window to match the coordinate plane below. Sketch the graphs of $f(x)$ and $g(x)$.

![Graph of Exponential Functions](image)

15. Should Alex revise his conjecture? Use the graph in Item 14b to explain.

16. a. Test Glenda’s conjecture by graphing $f(x) = 50x$ and $g(x) = 1.1^x$ on your graphing calculator. Do you agree with Glenda’s conjecture?

b. Now adjust your viewing window to match the coordinate plane below. Sketch the graphs of $f(x)$ and $g(x)$.

![Graph of Exponential Functions](image)

17. Should Glenda revise her conjecture? Use the graph in Item 16b to support your response.

18. Attend to precision. Is an exponential function always greater than a linear function? Explain your reasoning.
LESSON 22-3 PRACTICE

Isaac graphs \( f(x) = 9^x \) and \( g(x) = 0.5 \cdot 4^x \). His graphs are shown below.

21. Isaac states that \( g(x) \) will always be less than \( f(x) \). Explain Isaac's error.

22. Describe the relationship between the graphs of \( f(x) \) and \( g(x) \). Make a new graph to support your answer.

23. Make sense of problems. The math club has only 10 members and wants to increase its membership.
   - Julia proposes a goal of recruiting 2 new members each month. If the club meets this goal, the function \( y = 2x + 10 \) will give the total number of members \( y \) after \( x \) months.
   - Jorge proposes a goal to increase membership by 10% each month. If the club meets this goal, the function \( y = 10 \cdot 1.1^x \) will give the total number of members \( y \) after \( x \) months.

Club members want to choose the goal that will cause the membership to grow more quickly. Assume that the club will meet the recruitment goal that they choose. Which proposal should they choose? Use a graph to support your answer.
ACTIVITY 22 PRACTICE
Write your answers on notebook paper.
Show your work.

Lesson 22-1
In January of this year, a clothing store earned $175,000. Since then, earnings have increased by 10% each month. A function that models the store’s earnings after \( m \) months is \( e(m) = 175,000 \cdot (1.1)^m \). Use this information for Items 1–3.

1. Copy and complete the table.

<table>
<thead>
<tr>
<th>Months After January ((m))</th>
<th>Earnings (e(m))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$175,000</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

2. Make a graph of the function.

3. Predict the store’s earnings after 9 months.

4. A scientist studying a bacteria population recorded the data in the table below.

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Bacteria</td>
<td>8</td>
<td>20</td>
<td>50</td>
<td>125</td>
</tr>
</tbody>
</table>

Is the number of bacteria growing exponentially? Justify your response.

5. The function \( f(x) = 3 \cdot b^x \) is an exponential growth function. Which statement about the value of \( b \) is true?
   A. Because \( f(x) \) is an exponential growth function, \( b \) must be positive.
   B. Because \( f(x) \) is an exponential growth function, \( b \) must be greater than 1.
   C. Because \( f(x) \) is an exponential growth function, \( b \) must be between 0 and 1.
   D. The function represents exponential growth because \( 3 > 1 \), so \( b \) can have any value.

Lesson 22-2
A new car depreciates, or loses value, each year after it is purchased. A general rule is that a car loses 15% of its value each year.

Christopher bought a new car for $25,000. A function that models the value of Christopher’s car after \( t \) years is \( v(t) = 25,000 \cdot (0.85)^t \). Use this information for Items 6–8.

6. Copy and complete the table.

<table>
<thead>
<tr>
<th>Years After Purchase ((t))</th>
<th>Value of Car (v(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$25,000</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

7. Make a graph of the function.

8. Predict the value of Christopher’s car after 10 years.

For Items 9–12, graph each function and tell whether it represents exponential growth, exponential decay, or neither.

9. \( y = (2.5)^x \)
10. \( y = -0.75x \)
11. \( y = 3(1.5)^x \)
12. \( y = 80(0.25)^x \)

For Items 13–15, tell whether each function represents exponential growth, exponential decay, or neither. Justify your responses.

13. \[
\begin{array}{cccc}
    x & 0 & 1 & 2 & 3 \\
    y & 2 & 60 & 118 & 176 \\
\end{array}
\]

14. \[
\begin{array}{cccc}
    x & 0 & 1 & 2 & 3 \\
    y & 25 & 5 & 1 & 0.2 \\
\end{array}
\]
15. | x  | 0  | 2  | 4  | 6  |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>9</td>
<td>81</td>
<td>729</td>
</tr>
</tbody>
</table>

16. A wildlife biologist is studying an endangered species of salamander in a particular region. She finds the following data.

What was the initial number of salamanders in 2000?

17. Write a function that represents exponential decay. Explain how you know that your function represents exponential decay.

Lesson 22–3
Graph each of the following functions. Identify the values of $a$ and $b$, and describe how these values affect the graphs.

18. $y = -4(2)^x$
19. $y = -1(2.5)^x$
20. $y = 1.5(2)^x$
21. $y = 0.5(0.2)^x$

For Items 22–29, use a graphing calculator to graph each function. Compare each function to $a(x) = 2^x$, graphed below. Describe the similarities and differences between the graphs.

22. $f(x) = 0.5 \cdot 5^x$
23. $f(x) = 2 \cdot (1.1)^x$
24. $f(x) = 12^x$
25. $f(x) = 0.25 \cdot 4^x$
26. $f(x) = -3 \cdot 6^x$
27. $f(x) = -1 \cdot (0.3)^x$
28. $f(x) = 0.1 \cdot 2^x$
29. $f(x) = (0.5)^x$

30. Which function increases the fastest?
   A. $y = 104x$
   B. $y = -2 \cdot 15^x$
   C. $y = 12^x$
   D. $y = -220x$

31. Examine the graphs of $f(x) = 3^x$ and $g(x) = 5x$, shown below.

   a. Estimate the values of $x$ for which $f(x)$ is greater than $g(x)$.
   b. Estimate the values of $x$ for which $g(x)$ is greater than $f(x)$.
   c. As the values of $x$ decrease, the graph of $f(x)$ gets closer and closer to 0, or the $x$-axis. Will the graph ever intersect the $x$-axis? Explain.

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

32. Why can’t the value of $a$ in an exponential function be 0? Why can’t the value of $b$ be equal to 1?
Learning Target:
• Create an exponential function to model compound interest.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Predict and Confirm, Discussion Groups, Think-Pair-Share, Critique Reasoning

Madison received $10,000 in gift money when she graduated from college. She deposits the money into an account that pays 5% compound interest annually.

1. To find the total amount of money in her account after the first year, Madison must add the interest earned in the first year to the initial amount deposited.
   a. Calculate the earned interest for the first year by multiplying the amount of Madison’s deposit by the interest rate of 5%.

   b. Including interest, how much money did Madison have in her account at the end of the first year?

2. Madison wants to record the amount of money she will have in her account at the end of each year. Complete the table. Round amounts to the nearest cent.

<table>
<thead>
<tr>
<th>Year</th>
<th>Account Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$10,500.00</td>
</tr>
<tr>
<td>2</td>
<td>$11,025.00</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
The amount of money in the account increases by a constant growth factor each year.

3. Identify the constant growth factor to the nearest hundredth.

4. How is the interest rate on Madison's account related to the constant growth factor in Item 3?

5. Instead of calculating the amount of money in the account after each year, write an expression for each amount of money using $10,000 and repeated multiplication of the constant factor. Then rewrite each expression using exponents.

<table>
<thead>
<tr>
<th>Year</th>
<th>Account Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$10,000 \cdot 1.05 = $10,000 \cdot 1.05^1</td>
</tr>
<tr>
<td>2</td>
<td>($10,000 \cdot 1.05) \cdot 1.05 = $10,000 \cdot 1.05^2</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

6. Describe the relationship between the year number and the exponential expression.

7. Write an expression to represent the amount of money in the account at the end of Year 8.

8. Let $t$ equal the number of years.
   a. **Express regularity in repeated reasoning.** Write an expression to represent the amount of money in the account after $t$ years.

   b. Evaluate the expression for $t = 6$ to confirm that the expression is correct.

   c. Evaluate the expression for $t = 10$.

9. Write your expression as a function $m(t)$, where $m(t)$ is the total amount of money in Madison's account after $t$ years.
10. Use the data from the table in Item 2 to graph the function.

<table>
<thead>
<tr>
<th>Years</th>
<th>Money in Account</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5,000</td>
</tr>
<tr>
<td>4</td>
<td>10,000</td>
</tr>
<tr>
<td>6</td>
<td>15,000</td>
</tr>
<tr>
<td>8</td>
<td>20,000</td>
</tr>
<tr>
<td>10</td>
<td>25,000</td>
</tr>
<tr>
<td>12</td>
<td>30,000</td>
</tr>
</tbody>
</table>

11. Describe the function as linear or non-linear. Justify your response.

12. Identify the reasonable domain and range. Explain your reasoning.

13. Madison’s future plans include purchasing a home. She estimates that she will need at least $20,000 for a down payment. Determine the year in which Madison will have enough funds in her account for the down payment.

At the same time that Madison opens her account, her friend Frank deposits $10,000 in an account with an annual compound interest rate of 6%.

14. Write a new function to represent the total funds in Frank’s account, \( f(t) \), after \( t \) years.

15. Predict how the graph of Frank’s bank account balance will differ from the graph of Madison’s account balance.
16. Create a table of values for \( f(t) \), rounding to the nearest dollar. Then graph \( f(t) \) on the grid in Item 10. Confirm or revise your prediction in Item 15 using the table and graph.

<table>
<thead>
<tr>
<th>Year, ( t )</th>
<th>Funds in Frank's Account, ( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10,000</td>
</tr>
<tr>
<td>1</td>
<td>$10,600</td>
</tr>
<tr>
<td>2</td>
<td>$11,236</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

At the same time that Madison and Frank open their accounts, another friend, Kasey, opens a savings account in a different bank. Kasey deposits $12,000 at an annual compound interest rate of 4%.

17. How does Kasey’s situation change the function? Write a new function \( k(t) \) to represent Kasey’s account balance at any year \( t \).

18. **Critique the reasoning of others.** Kasey believes that since she started her account with more money than Madison or Frank, she will always have more money in her account than either of them, even though her interest rate is lower. Is Kasey correct? Justify your response using a table, graph, or both.
Lesson 23-1
Compound Interest

19. Over a long period of time, does the initial deposit or the interest rate have a greater effect on the amount of money in an account that has interest compounded yearly? Explain your reasoning.

Most savings institutions offer compounding intervals other than annual compounding. For example, a bank that offers quarterly compounding computes interest on an account every quarter; that is, at the end of every 3 months. Instead of computing the interest once each year, interest is computed four times each year. If a bank advertises that it is offering 8% interest compounded quarterly, 8% is not the actual growth factor. Instead, the bank will use \( \frac{8\%}{4} = 2\% \) to determine the quarterly growth factor.

20. What is the quarterly interest rate for an account with an annual interest rate of 5%, compounded quarterly?

21. Suppose that Madison invested her $10,000 in the account described in Item 20.
   a. In the table below, determine Madison’s account balance after the specified times since her initial deposit.

<table>
<thead>
<tr>
<th>Time Since Initial Deposit</th>
<th>Number of Times Interest Has Been Compounded</th>
<th>Account Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 months</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 months</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 months</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 year</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 years</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t ) years</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b. Write a function \( A(t) \) to represent the balance in Madison’s account after \( t \) years.

   c. Calculate the balance in Madison’s account after 20 years.
22. For the compounding periods given below, write a function to represent the balance in Madison's account after $t$ years. Then calculate the balance in the account after 20 years. She is investing $10,000 at a rate of 5% annual compound interest.

   a. Yearly:
   
   b. Quarterly:
   
   c. Monthly:
   
   d. Daily (assume there are 365 days in a year):

23. What is the effect of the compounding period on the amount of money in the account after 20 years as the number of times the interest is compounded each year increases?

24. Write a function that gives the amount of money in Frank's account after $t$ years when 6% annual interest is compounded monthly.

25. Create a table and a graph for the function in Item 24. Be sure to label the units on the $x$-axis correctly.

26. Create a table showing the amount of money in Nick's account after 0–8 years.

27. Write a function that gives the amount of money in Nick's account after $t$ years. Identify the reasonable domain and range.

28. Create a graph of your function.

29. Explain how Nick's account balance would be different if he deposited his money into an account that pays 2% annual interest, compounded annually. Graph this situation on the same coordinate plane that you used in Item 28. Describe the similarities and differences between the graphs.
Lesson 23-2
Population Growth

Learning Targets:
• Create an exponential function to fit population data.
• Interpret values in an exponential function.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Predict and Confirm, Think Aloud, Sharing and Responding, Construct Arguments

The population of Nevada since 1950 is shown in the table in the My Notes section.

1. Graph the data from the table.

2. Use the table and the graph to explain why the data are not linear.

3. a. Complete the table by finding the approximate ratio between the populations in each decade.

<table>
<thead>
<tr>
<th>Decades Since 1950</th>
<th>Resident Population</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>160,083</td>
<td>--</td>
</tr>
<tr>
<td>1</td>
<td>285,278</td>
<td>285,278/160,083 ≈ 1.782</td>
</tr>
<tr>
<td>2</td>
<td>488,738</td>
<td>488,738/285,278 ≈</td>
</tr>
<tr>
<td>3</td>
<td>800,508</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1,201,833</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1,998,257</td>
<td></td>
</tr>
</tbody>
</table>

b. Explain how the table shows that the data are not exponential.

The data are not exactly exponential, but the shape of the graph resembles an exponential curve. Also, the table in Item 3a shows a near-constant factor. These suggest that the data are approximately exponential. Use exponential regression to find an exponential function that models the data.

MATH TERMS
Exponential regression is a method used to find an exponential function that models a set of data.
4. Use a graphing calculator to determine the exponential regression equation to model the relationship between the decades since 1950 and the population.
   a. The calculator returns two values, \( a \) and \( b \). Write these values below. Round \( a \) to the nearest whole number and \( b \) to the nearest thousandth, if necessary.
      \[
      a = \quad b =
      \]
   b. The general form of an exponential function is \( y = ab^x \). Use this general form and the values of \( a \) and \( b \) from Part (a) to write an exponential function that models Nevada’s population growth.

5. **Reason abstractly.** What does the value of \( b \) tell you about Nevada’s population growth?

6. Interpret the value of \( a \) in terms of Nevada’s population. How is this value related to the graph?

7. What do the domain values represent?

8. What would the \( x \)-intercept represent in terms of Nevada’s population? Does the graph have an \( x \)-intercept? Explain.
Lesson 23-2
Population Growth

9. **Make sense of problems.** Describe how to estimate the population of Nevada in 1995 using each of the following:
   a. the function identified in Item 4b
   b. the graph of the function
   c. a table

10. Estimate the population in 1995. Which method did you use, and why?

11. a. Estimate Nevada’s population in 2010.
   
   b. **Construct viable arguments.** Which estimate do you think is likely to be more accurate, your estimate of the population in 1995 or in 2010? Explain.

   The function for the growth rate of Nevada’s population estimates the growth per decade. You can use this rate to estimate the growth per year, or the annual growth rate.

12. Let \( n \) be the number of years since 1950. Write an equation that gives the number of years \( n \) in \( x \) decades. Solve your equation for \( x \).

13. Rewrite the function that models Nevada’s population from Item 4b. Then write the function again, but replace \( x \) with the equivalent expression for \( x \) from Item 12.

14. Simplify to write the function in the form \( y = ab^n \).

   \[
   y = 170,377 \cdot (1.645)^{\frac{n}{10}} = 170,377 \cdot \left(1.645{\frac{1}{10}}\right)^{n/10} \\
   \approx 170,377 \cdot ( )^n
   \]
15. What is the approximate annual growth rate of Nevada’s population? How do you know?

16. To find the approximate population of Nevada in 2013, what value should you use for \( n \)? Explain.

17. Use the function from Item 14 to find the approximate population of Nevada for the year 2013.

18. Compare the approximate population for 2013 that you found in Item 17 to the approximate population you found for 2010 in Item 11. Does your estimate for 2013 seem reasonable? Why or why not?

19. Create a graph showing the annual growth of Nevada’s population.

20. Describe the similarities and differences between the graph in Item 19 and the previous graph of Nevada’s population from Item 4c.

**Check Your Understanding**

19. Create a graph showing the annual growth of Nevada’s population.

20. Describe the similarities and differences between the graph in Item 19 and the previous graph of Nevada’s population from Item 4c.

**LESSON 23-2 PRACTICE**

The population of Texas from 1950 to 2000 is shown in the table below.

<table>
<thead>
<tr>
<th>Year</th>
<th>Resident Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1950</td>
<td>7,711,194</td>
</tr>
<tr>
<td>1960</td>
<td>9,579,677</td>
</tr>
<tr>
<td>1970</td>
<td>11,198,655</td>
</tr>
<tr>
<td>1980</td>
<td>14,225,513</td>
</tr>
<tr>
<td>1990</td>
<td>16,986,510</td>
</tr>
<tr>
<td>2000</td>
<td>20,851,820</td>
</tr>
</tbody>
</table>

21. **Use appropriate tools strategically.** Use a graphing calculator to find a function that models Texas’s population growth.

22. Create a graph showing the actual population from the table and the approximate population from the function in Item 21.

23. Is the function a good fit for the data? Why or why not?

24. Describe the meanings of the domain, range, \( y \)-intercept, and \( x \)-intercept in the context of Texas’s population growth.
**ACTIVITY 23 PRACTICE**

Write your answers on notebook paper. Show your work.

**Lesson 23-1**

1. Four friends deposited money into savings accounts. The amount of money in each account is given by the functions below.
   
   Marisol: \( m(t) = 100 \cdot (1.01)^t \)
   
   Iris: \( i(t) = 200 \cdot (1.04)^t \)
   
   Brenda: \( b(t) = 300 \cdot (1.05)^t \)
   
   José: \( j(t) = 400 \cdot (1.03)^t \)

   Which statement is correct?
   
   A. José has the greatest interest rate.
   
   B. Brenda has the greatest initial deposit.
   
   C. The person with the least initial deposit also has the least interest rate.
   
   D. The person with the greatest initial deposit also has the greatest interest rate.

Darius makes an initial deposit into a bank account, and then earns interest on his account. He records the amount of money in his account each year in the table below. Use this table for Items 2–5.

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$4000.00</td>
</tr>
<tr>
<td>1</td>
<td>$4120.00</td>
</tr>
<tr>
<td>2</td>
<td>$4243.60</td>
</tr>
<tr>
<td>3</td>
<td>$4370.91</td>
</tr>
<tr>
<td>4</td>
<td>$4502.04</td>
</tr>
</tbody>
</table>

2. Make a graph showing the amount of money in Darius’s account each year.

3. Identify the constant factor. Round to the nearest hundredth.

4. Identify the reasonable domain and range. Explain your answers.

5. What is the annual interest rate? How do you know?

The amount of money \( y \) in Jesse’s checking account \( t \) years after the account was opened is given by the function \( j(t) = 15,000 \cdot (1.02)^t \). Use this information for Items 6–10.

6. What was the initial amount of money deposited in Jesse’s account?

7. What is the annual interest rate?

8. Create a graph of the amount of money in Jesse’s checking account.

9. Interpret the meaning of the \( y \)-intercept in the context of Jesse’s account.

10. Find the amount of money in the account after 4 years.

The two graphs on the coordinate grid below represent the amounts of money in two different savings accounts. Graph \( a \) represents the amount of money in Allison’s account, and graph \( b \) represents the amount of money in Boris’s account. Use the graph for Items 11–13.

11. Whose account had a higher initial deposit? Use the graph to justify your answer.

12. What was the amount of Allison’s initial deposit?

13. Identify the reasonable domain and range for each function, and explain your answers.
Maria's bank offers two types of savings accounts. The first has an annual interest rate of 8% compounded annually. The second also has an annual interest rate of 8%, but it is compounded monthly. She is going to open an account by depositing $1000. Use this information for Items 14–19.

14. If Maria chooses the first account, determine the amount of money she will have in the account after 3 years.

15. Write a function that gives the amount of money in the first account after \( t \) years.

16. Write a function that gives the amount of money in the second account after \( t \) years.

17. What is the monthly interest rate for the second account?

18. If Maria chooses the second account, determine the amount of money she will have in the account after 1 year.

19. After 10 years, which account will have the higher balance?

Lesson 23-2

20. Which function is the best model for the data in the table?

\[
\begin{array}{c|c|c|c|c|c}
  x & 0 & 3 & 6 & 9 & 4 \\
  y & 16.5 & 12 & 8.5 & 4.6 & 0 \\
\end{array}
\]

22. The head circumference of an infant is measured and recorded to track the infant's growth and development. Nathan's head circumferences from age 3 months through 12 months are recorded in the table below. Use the table for Items 23–27.

<table>
<thead>
<tr>
<th>Age (months)</th>
<th>Head Circumference (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>43</td>
</tr>
<tr>
<td>4</td>
<td>44</td>
</tr>
<tr>
<td>5</td>
<td>44.7</td>
</tr>
<tr>
<td>6</td>
<td>45.2</td>
</tr>
<tr>
<td>7</td>
<td>45.8</td>
</tr>
<tr>
<td>8</td>
<td>46.3</td>
</tr>
<tr>
<td>9</td>
<td>47.1</td>
</tr>
<tr>
<td>10</td>
<td>47.6</td>
</tr>
<tr>
<td>11</td>
<td>48.0</td>
</tr>
<tr>
<td>12</td>
<td>48.3</td>
</tr>
</tbody>
</table>

23. Use a graphing calculator to find an exponential function to approximate Nathan's head circumference.

24. Identify the reasonable domain and range for the function in Item 23. Explain your answers.

25. Create a graph showing Nathan's head circumference.

26. Determine the growth rate, and explain how you found your answer.

27. Interpret the meaning of the \( y \)-intercept in the context of Nathan's head circumference.

MATHEMATICAL PRACTICES

Attend to Precision

28. Explain why the \( x \)-intercept can have a meaning in the context of a situation, such as population growth, but cannot be shown on the graph.
Mr. Davis has just become a grandfather! He wants to invest money for his new granddaughter’s college education.

Mr. Davis has done some research on savings bonds. He has learned that you buy a savings bond from the government or from a bank. After one year, you can cash in your bond and get back the money you paid for it. However, if you wait at least five years, you will get back your money plus interest.

Mr. Davis has also learned that he can buy paper bonds or electronic bonds. While there are many similarities and differences between the bonds, Mr. Davis has summarized the most important information below.

<table>
<thead>
<tr>
<th>Paper Bond</th>
<th>Electronic Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current rate of interest: 1.8% annual, but the interest rate may change over the life of the bond.</td>
<td>Current rate of interest: 1.4% annual; this rate will not change.</td>
</tr>
<tr>
<td>For both bonds, the interest is compounded semiannually (every 6 months, or twice per year) for 30 years or until the bond is cashed in, whichever comes first.</td>
<td></td>
</tr>
</tbody>
</table>

Mr. Davis decides to buy a $5000 bond that he will give to his granddaughter on her 18th birthday. He will use the current interest rates to decide which bond he will purchase.

1. Using the current interest rate, a function that gives the value of a $5000 paper bond after $t$ years is $p(t) = 5000 \cdot (1.009)^{2t}$.
   a. How is the interest rate of 1.8% related to the function?
   b. Why is the exponent $2t$ instead of $t$?
   c. Use the function to determine the value of the bond in 18 years. Round your answer to the nearest cent.

2. a. Write a function $e(t)$ that gives the value of a $5000 electronic bond after $t$ years.
   b. Use your function to determine the value of the bond in 18 years. Round your answer to the nearest cent.

3. Identify the reasonable domain and range for each function. Explain your answers.

4. Use a graphing calculator to graph both functions on the same coordinate plane. Sketch the graphs and label each function.

5. Explain to Mr. Davis which bond you think he should purchase and why.

6. Mr. Davis’s accountant has more information about electronic bonds. She tells Mr. Davis that if you keep an electronic bond for 20 years, the value becomes double what you paid for it. Would this change your advice to Mr. Davis? Explain.
### Scoring Guide

<table>
<thead>
<tr>
<th>Mathematics Knowledge and Thinking (Items 1c, 2b)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Effective understanding of and accuracy in evaluating an exponential function</td>
<td>• Largely correct understanding of and accuracy in evaluating an exponential function</td>
<td>• Partial understanding of and some difficulty in evaluating an exponential function</td>
<td>• Incomplete understanding of and significant difficulty in evaluating an exponential function</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem Solving (Items 5, 6)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Appropriate and efficient strategy that results in a correct answer</td>
<td>• Strategy that may include unnecessary steps but results in a correct answer</td>
<td>• Strategy that results in some incorrect answers</td>
<td>• No clear strategy when solving problems</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mathematical Modeling/ Representations (Item 1a, 1b, 2a, 3, 4)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Fluency in representing a real-world scenario using an exponential function, including identification of a reasonable domain and range</td>
<td>• Adequate understanding of how to represent a real-world scenario using an exponential function, including identification of a reasonable domain and range</td>
<td>• Partial understanding of how to represent a real-world scenario using an exponential function, including identification of a reasonable domain and range</td>
<td>• Inaccurate or incomplete understanding of how to represent a real-world scenario using an exponential function, including identification of a reasonable domain and range</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate graphs of exponential functions</td>
<td>• Little difficulty graphing exponential functions</td>
<td>• Partially accurate graphs of exponential functions</td>
<td>• Inaccurate or incomplete graphs of exponential functions</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reasoning and Communication (Items 5, 6)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Precise use of appropriate math terms and language to make and justify a recommendation</td>
<td>• Recommendation with an adequate justification</td>
<td>• Recommendation with a misleading or confusing justification</td>
<td>• No recommendation or a recommendation with an inaccurate or incomplete justification</td>
<td></td>
</tr>
</tbody>
</table>
A solar panel is a device that collects and converts solar energy into electricity or heat. The solar panel consists of interconnected solar cells. The panels can have differing numbers of solar cells and can come in square or rectangular shapes.

1. How many solar cells are in the panel below?

2. **Reason abstractly.** If a solar panel has four rows as the picture does, but can be extended to have an unknown number of columns, \(x\), write an expression to give the number of solar cells that could be in the panel.

3. Write an expression that would give the total number of cells in the panel for a solar panel having \(x\) rows and \(x\) columns.

4. If there were 5 panels like those found in Item 3, write an expression to represent the total number of solar cells.

All the answers in Items 1–4 are called **terms**. A **term** is a number, variable, or the product of a number and/or variable(s).

5. Write an expression to represent the sum of your answers from Items 1, 2, and 4.
Expressions like the answer to Item 5 are called polynomials. A **polynomial** is a single term or the sum of two or more terms with **whole-number powers**.

6. List the terms of the polynomial you wrote in Item 5.

7. What are the **coefficients** of the polynomial in Item 5? What is the **constant term**?

8. Tell whether each expression is a polynomial. Explain your reasoning.
   - a. $3x^{-2} - 5$
   - b. $6x + 4x^2$
   - c. 15
   - d. $2 + x^{\frac{1}{2}}$

9. For the expressions in Item 8 that are polynomials, identify the terms, coefficients, and constant terms.

The **degree of a term** is the sum of the exponents of the variables contained in the term.

10. Identify the degree and coefficient of each term in the polynomial $4x^5 + 12x^3 + x^2 - x + 5$.

<table>
<thead>
<tr>
<th>Term</th>
<th>Degree</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4x^5$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$12x^3$</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>$x^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-x$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

11. Make use of structure. For the polynomial $2x^2y - 6x^2y^2 + 9xy - 13y^5 + 5x + 15$, list each term and identify its degree and coefficient. Identify the constant term.
Lesson 24-1
Polynomial Terminology

The **degree of a polynomial** is the greatest degree of any term in the polynomial.

12. Identify the degree and constant term of each polynomial.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree of Polynomial</th>
<th>Constant Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x^2 + 3x + 7$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$-5y^3 + 4y^2 - 8y - 3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$36 + 12x + x^2$</td>
<td>36</td>
<td></td>
</tr>
</tbody>
</table>

The **standard form of a polynomial** is a polynomial whose terms are written in **descending order** of degree. The **leading coefficient** is the coefficient of a polynomial’s leading term when the polynomial is written in standard form.

A polynomial can be classified by the number of terms it has when it is in simplest form.

<table>
<thead>
<tr>
<th>Name</th>
<th>Number of Terms $n$</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>monomial</td>
<td>1</td>
<td>$8 \text{ or } -2x \text{ or } 3x^2$</td>
</tr>
<tr>
<td>binomial</td>
<td>2</td>
<td>$3x + 2 \text{ or } 4x^2 - 7x$</td>
</tr>
<tr>
<td>trinomial</td>
<td>3</td>
<td>$-x^2 - 3x + 9$</td>
</tr>
<tr>
<td>polynomial</td>
<td>$n &gt; 3$</td>
<td>$9x^4 - x^3 - 3x^2 + 7x - 2$</td>
</tr>
</tbody>
</table>

**MATH TERMS**

**Descending order** of degree means that the term that has the highest degree is written first, the term with the next highest degree is written next, and so on.

**READING MATH**

The prefixes mono (one), bi (two), tri (three), and poly (many) appear in many math terms such as bisect (cut in half), triangle (three-sided figure), and polygon (many-sided figure).
13. Fill in the missing information in the table below.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Number of Terms</th>
<th>Name</th>
<th>Leading Coefficient</th>
<th>Constant Term</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x^2 - 5x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-2x^2 + 13x + 6$</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$15x^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$5p^3 + 2p^2 - p - 7$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a^2 - 25$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.23x^3 + 0.54x^2 - 0.58x + 0.0218$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-9.8t^2 - 20t + 150$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

14. Is the following statement true or false? Explain.
   “All polynomials are binomials.”

15. Describe your first step for writing $3x - 5x^2 + 7$ in standard form.

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**Check Your Understanding**

14. Is the following statement true or false? Explain.
   “All polynomials are binomials.”

15. Describe your first step for writing $3x - 5x^2 + 7$ in standard form.

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**LESSON 24-1 PRACTICE**

For Items 16–20, use the polynomial $4x^3 + 3x^2 - 9x + 7$.

16. Name the coefficients of the terms in the polynomial that have variables.
17. List the terms, and give the degree of each term.
18. What is the degree of the polynomial?
19. Identify the leading coefficient of the polynomial.
20. Identify the constant term of the polynomial.

Write each polynomial in standard form.

21. $9 + 8x^2 + 2x^3$  
22. $y^2 + 1 + 4y^3 - 2x$

23. **Construct viable arguments.** Is the expression $5x^2 + \sqrt{2}x$ a polynomial? Justify your response.
Learning Targets:
- Use algebra tiles to add polynomials.
- Add polynomials algebraically.

SUGGESTED LEARNING STRATEGIES: Discussion Groups, Use Manipulatives, Create Representations, Close Reading, Note Taking

Notice that in the solar panels at the right, there are \(4^2\) or 16 cells. Each column has 4 cells.

1. If a square solar panel with an unknown number of cells along the edge can be represented by \(x^2\), how many cells would be in one column of the panel?

A square solar panel with \(x\) rows and \(x\) columns can be represented by the algebra tile:

![Algebra tile representation of \(x^2\)](image)

A column of \(x\) cells can be represented by using the tile \(x\), and a single solar cell can be represented by \([+1]\).

Suppose there were 3 square solar panels that each had \(x\) columns and \(x\) rows, 2 columns with \(x\) cells, and 3 single solar cells. You can represent \(3x^2 + 2x + 3\) using algebra tiles.

![Algebra tile representation of \(3x^2 + 2x + 3\)](image)

2. Represent \(2x^2 - 3x + 2\) using algebra tiles. Draw a picture of the representation below.

MATH TIP
The additive inverse of the \(x^2\), \(x\), and 1 algebra tiles can be represented with another color, or the flip side of the tile.
Adding polynomials using algebra tiles can be done by:
- modeling each polynomial
- identifying and removing zero pairs
- writing the new polynomial

**Example A**

Add \((3x^2 - 3x - 5) + (2x^2 + 5x + 3)\) using algebra tiles.

**Step 1:** Model the polynomials.

\[
\begin{align*}
3x^2 &- 3x - 5 \\
2x^2 &+ 5x + 3
\end{align*}
\]

**Step 2:** Identify and remove zero pairs.

**Step 3:** Combine like tiles.

**Step 4:** Write the polynomial for the model in Step 3.

\[5x^2 + 2x - 2\]

**Solution:** \((3x^2 - 3x - 5) + (2x^2 + 5x + 3) = 5x^2 + 2x - 2\)

**Try These A**

Add using algebra tiles.

a. \((x - 2) + (2x + 5)\)

b. \((2y^2 + 3y + 6) + (3y^2 - 4)\)

c. \((2x^2 + 3x + 9) + (-x^2 - 4x - 6)\)

d. \((5 - 3x + x^2) + (2x + 4 - 3x^2)\)
Lesson 24-2
Adding Polynomials

3. Use appropriate tools strategically. Can you use algebra tiles to add \((4x^4 + 3x^2 + 15) + (x^4 + 10x^3 - 4x^2 + 22x - 23)\)? If so, model the polynomials and add. If not, explain why.

**Like terms** in an expression are terms that have the same variable and exponent for that variable. All constants are like terms.

4. State whether the terms are like or unlike terms. Explain.

   a. \(2x; 2x^3\)

   b. \(5; 5x\)

   c. \(-3y; 3y\)

   d. \(x^2y; xy^2\)

   e. \(14; -0.6\)

5. Attend to precision. Using vocabulary from this activity, describe a method that could be used to add polynomials without using algebra tiles.
Use the properties of real numbers to add polynomials algebraically.

**Example B**
Add \((3x^3 + 2x^2 - 5x + 7) + (4x^3 + 2x - 3)\) horizontally and vertically. Write your answer in standard form.

**Horizontally**

**Step 1:** Identify like terms. \((3x^3 + 2x^2 - 5x + 7) + (4x^3 + 2x - 3)\)

**Step 2:** Group like terms. \((3x^3 + 4x^3) + (2x^2) + (-5x + 2x) + (7 - 3)\)

**Step 3:** Add the coefficients of like terms. \(7x^3 + 2x^2 - 3x + 4\)

**Solution:** \((3x^3 + 2x^2 - 5x + 7) + (4x^3 + 2x - 3) = 7x^3 + 2x^2 - 3x + 4\)

**Vertically**

**Step 1:** Vertically align like terms. \(3x^3 + 2x^2 - 5x + 7\)

**Step 2:** Add the coefficients of like terms. \(\frac{4x^3 + 2x}{7x^3 + 2x^2 - 3x + 4}\)

**Solution:** \((3x^3 + 2x^2 - 5x + 7) + (4x^2 + 2x - 3) = 7x^3 + 2x^2 - 3x + 4\)

**Try These B**
Add. Write your answers in standard form.

a. \((4x^2 + 3) + (x^2 - 3x + 5)\)  
b. \((10y^2 + 8y + 6) + (17y^2 - 11)\)

c. \((9x^2 + 15x + 21) + (-13x^2 - 11x - 26)\)

6. Are the answers to Try These B polynomials? Justify your response.

7. Explain why the sum of two polynomials will always be a polynomial.
Lesson 24-2
Adding Polynomials

LESSON 24-2 PRACTICE
Add. Write your answers in standard form.

12. \((3x^2 + x + 5) + (2x^2 + x - 5)\)
13. \((-4x^2 + 2x - 1) + (x^2 - x + 9)\)
14. \((7x^2 - 2x + 3) + (3x^2 + 2x + 7)\)
15. \((-x^3 + 5x + 2) + (-3x^2 + x - 9)\)

Write the perimeter of each figure as a polynomial in standard form.

16. \(3x - 1\)
17. \(4x - 7\)

18. **Critique the reasoning of others.** A student added the expressions \(x^4 + 5x^2 - 2x + 1\) and \(2x^4 + x^3 + 2x - 7\). Identify and correct the student's error.

\[
\begin{align*}
\text{Corrected Expression:} & \quad x^4 + 5x^2 - 2x + 1 \\
\text{Correction:} & \quad 2x^4 + x^3 + 2x - 7 \\
\text{Result:} & \quad 3x^4 + 6x^2 - 6
\end{align*}
\]
Learning Target:
- Subtract polynomials algebraically.

SUGGESTED LEARNING STRATEGIES: Note Taking, Close Reading, Think-Pair-Share

To subtract a polynomial you add its **opposite**, or subtract each of its terms.

Example A
Subtract \((2x^3 + 8x^2 + x + 10) - (5x^2 - 4x + 6)\) horizontally and vertically. Write the answer in standard form.

**Horizontally**

Step 1: Distribute the negative. \((2x^3 + 8x^2 + x + 10) - (5x^2 - 4x + 6)\)

Step 2: Identify like terms. \(2x^3 + 8x^2 + x + 10 - 5x^2 + 4x - 6\)

Step 3: Group like terms. \(2x^3 + (8x^2 - 5x^2) + (x + 4x) + (10 - 6)\)

Step 4: Combine coefficients of like terms. \(2x^3 + 3x^2 + 5x + 4\)

Solution: \((2x^3 + 8x^2 + x + 10) - (5x^2 - 4x + 6) = 2x^3 + 3x^2 + 5x + 4\)

**Vertically**

Step 1: Vertically align like terms. \[
\begin{align*}
2x^3 + 8x^2 + x + 10 \\
- (5x^2 - 4x + 6)
\end{align*}
\]

Step 2: Distribute the negative. \[
\begin{align*}
2x^3 + 8x^2 + x + 10 \\
- 5x^2 + 4x - 6
\end{align*}
\]

Step 3: Combine coefficients of like terms. \(2x^3 + 3x^2 + 5x + 4\)

Solution: \((2x^3 + 8x^2 + x + 10) - (5x^2 - 4x + 6) = 2x^3 + 3x^2 + 5x + 4\)
Lesson 24-3
Subtracting Polynomials

Try These A
Subtract. Write your answers in standard form.

a. \((5x - 5) - (x + 7)\)

b. \((2x^2 + 3x + 2) - (-5x^2 - 2x - 9)\)

c. \((y^2 + 3y + 8) - (4y^2 - 9)\)

d. \((12 + 5x + 14x^2) - (8x + 15 - 7x^2)\)

1. Are the answers to Try These A polynomials? Justify your response.

2. Explain why the difference of two polynomials will always be a polynomial.

Polynomials are closed under subtraction. A set is closed under subtraction if the difference of any two elements in the set is also an element of the set.
Rewrite each difference as addition of the opposite, or additive inverse, of the second polynomial.

3. \((x^2 + 2x + 3) - (4x^2 - x + 5)\)
4. \((5y^2 + y - 2) - (-y^2 - 3y + 4)\)
5. **Critique the reasoning of others.** Gil used the vertical method to subtract \((3x^2 - 5x + 2) - (x^2 + 2x + 4)\) as shown below. Identify Gil’s error.

\[
\begin{array}{c}
3x^2 - 5x + 2 \\
- x^2 + 2x + 4 \\
\hline
2x^2 - 3x + 6
\end{array}
\]

**LESSON 24-3 PRACTICE**

Subtract. Write your answers in standard form.

6. \((2x^2 + 4x + 1) - (7x^2 - 3x - 4)\)
7. \((x^2 + 3x - 9) - (x^2 + 2x - 8)\)
8. \((9x^2 + x - 12) - (14x^2 - 7x - 2)\)
9. \((x^2 + 3x - 6) - (5x - 6)\)
10. \((y^4 + y^2 + 2y) - (-y^4 + 3)\)
11. Write two polynomials whose difference is \(6x + 3\).
12. **Model with mathematics.** A rectangular piece of paper has area \(4x^2 + 3x + 2\). A square is cut from the rectangle and the remainder of the rectangle is discarded. The area of the discarded paper is \(3x^2 + x + 1\). What is the area of the square?
ACTIVITY 24 PRACTICE
Write your answers on notebook paper. 
Show your work.

Lesson 24-1
For Items 1–5, use the polynomial 
$5x^4 - 2x^2 + 8x - 3$.

1. Identify the coefficients of the variable terms of 
the polynomial.
2. List the terms, and give the degree of each term.
3. State the degree of the polynomial.
4. Identify the leading coefficient of the polynomial.
5. Identify the constant term of the polynomial.
6. Consider the expressions $3x^2 + 2x - 7$ and 
$3x^2 + 2x - 7x^0$. Are the expressions equivalent? 
Explain.

Write each polynomial in standard form.
7. $5x^2 - 2x^3 - 10$
8. $11 + y^2 - 8y$
9. $y^3 - 12 - y^3 + y^4$
10. $9x + 7 - 5x^3$

Lesson 24-2
Add. Write your answers in standard form.
11. $(4x + 9) + (3x - 5)$
12. $(2x^2 + 3x - 1) + (x^2 - 5x + 2)$
13. $(x^3 + 5x^2 + 3) + (2x^3 - 5x^2)$
14. $(7y^3 - 2y^2 + 5) + (4y^3 - 3y)$

15. Which expression represents the perimeter of the 
trapezoid?

A. $11x + 4$
B. $4x + 2$
C. $15x + 6$
D. $13x + 8$

16. The length of each side of a square is $4y + 5$. 
Draw and label the square, and write an 
expression to represent its perimeter.

Lesson 24-3
17. Which expression is equivalent to 
$10x - (7x - 1)$? 
A. $3x - 1$
B. $3x + 1$
C. $17x - 1$
D. $17x + 1$

Subtract. Write your answers in standard form.
18. $(5x - 4) - (3x + 2)$
19. $(3x^2 - 2x + 7) - (2x^2 + 2x - 7)$
20. $(8y^2 - 3y + 6) - (-2y^2 - 3)$
21. $(x^2 - 5x) - (4x - 6)$
Determine the sum or difference. Write your answers in standard form.

22. \((5y^2 + 3y + 7) + (7y - 2)\)
23. \((3x^2 + x + 9) - (2x^2 + x + 2)\)
24. \((x^3 + 3x^2 + 12) - (5x^3 + 7x^2)\)
25. \((8x^3 - 5x + 7) + (4x^3 + 3x^2 - 3x - 4)\)
26. \((-4y^3 - 2y + 1) - (7y^3 + y^2 - y + 5)\)
27. \((3x + 7y) + (-4x + 3y)\)
28. \((5x^2 + 8xy + y^2) - (-x^2 + 4xy - 5y^2)\)

29. A playground has a sidewalk border around a play area.

The total area of the playground, the larger rectangle, is \(16x^2 - 5x + 2\). The area of the play area, the smaller rectangle, is \(10x^2 + 3x - 1\). Write an expression to represent the area of the sidewalk.

30. To make a box, four corners of a rectangular piece of cardboard are cut out and the box is folded and taped.

The area of the cardboard, after the corners are cut out, is \(28x^2 + 12x + 32\). The area of each cut-out corner is \(2x^2 + 3\). Write an expression to represent the area of the original piece of cardboard.

**MATHEMATICAL PRACTICES**

**Reason Abstractly and Quantitatively**

31. The set of polynomials is closed under the operations of addition and subtraction. This means that when you add or subtract two polynomials, the result is also a polynomial.
   
a. Are the integers closed under addition and subtraction? In other words, when you add or subtract two integers, is the result always an integer? Justify your response.

b. Give a counterexample to show that the whole numbers are **not** closed under subtraction.
Learning Targets:
• Use a graphic organizer to multiply expressions.
• Use the Distributive Property to multiply expressions.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Look for a Pattern, Discussion Groups, Create Representations, Graphic Organizer

Tri-Com Computers is a company that sets up local area networks in offices. In setting up a network, consultants need to consider not only where to place computers, but also where to place peripheral equipment, such as printers.

Tri-Com typically sets up local area networks of computers and printers in square or rectangular offices. Printers are placed in each corner of the room. The primary printer A serves the first 25 computers and the other three printers, B, C, and D, are assigned to other regions in the room. Below is an example.

1. If each dot represents a computer, how many computers in this room will be assigned to each of the printers?

2. What is the total number of computers in the room? Describe two ways to find the total.
Another example of an office in which Tri-Com installed a network had 9 computers along each wall. The computers are aligned in an array with the number of computers in each region determined by the number of computers along the wall.

3. A technician claimed that since $9 = 5 + 4$, the number of computers in the office could be written as an expression using only the numbers 5 and 4. Is the technician correct? Explain.

4. Show another way to determine the total number of computers in the office.

5. Rewrite the expression $(5 + 4)(5 + 4)$ using the Distributive Property.

6. Make sense of problems. Explain why $(5 + 4)(5 + 4)$ could be used to determine the total number of computers.
Lesson 25-1
Multiplying Binomials

7. The office to the right has $8^2$ computers. Fill in the number of computers in each section if it is split into a $(5 + 3)^2$ configuration.

8. What is the total number of computers? Describe two ways to find the total.

\[
\begin{array}{c|c|c|c}
5 & 8 & 3 \\
\hline
5 & & \\
\hline
& & 8 \\
\hline
3 & & \\
\end{array}
\]

9. For each possible office configuration below, draw a diagram like the one next to Item 7. Label the number of computers on the edge of each section and determine the total number of computers in the room by adding the number of computers in each section.

a. $(2 + 3)^2$  

b. $(4 + 1)^2$  

c. $(3 + 7)^2$
Tri-Com has a minimum requirement of 25 computers per installation arranged in a 5 by 5 array. Some rooms are larger than others and can accommodate more than 5 computers along each wall to complete a square array. Use a variable expression to represent the total number of computers needed for any office having \(x\) more than the 5 computer minimum along each wall.

10. One technician said that \(5^2 + x^2\) would be the correct way to represent the total number of computers in the office space. Use the diagram to explain how the statement is incorrect.

11. **Model with mathematics.** Write an expression for the sum of the number of computers in each region in Item 10.

12. For each of the possible room configurations, determine the total number of computers in the room.
   a. \((2 + x)^2\)
   b. \((x + 3)^2\)
   c. \((x + 6)^2\)
The graphic organizer below can be used to help arrange the multiplications of the Distributive Property. It does not need to be related to the number of computers in an office. For example, this graphic organizer shows $5 \cdot 7 = (3 + 2)(4 + 3)$.

13. Draw a graphic organizer to represent the expression $(5 + 2)(2 + 3)$. Label each inner rectangle and find the sum.

14. Draw a graphic organizer to represent the expression $(6 - 3)(4 - 2)$. Label each inner rectangle and find the sum.

15. Multiply the binomials in Item 14 using the Distributive Property. What do you notice?
You can use the same graphic organizer to multiply binomials that contain variables. The following diagram represents \((x - 2)(x - 3)\).

16. Use the graphic organizer above to represent the expression \((x - 2)(x - 3)\). Label each inner rectangle and find the sum.

17. Multiply the binomials in Item 16 using the Distributive Property. What do you notice?

18. Determine the product of the binomials.
   a. \((x - 7)(x - 5)\)  
   b. \((x - 7)(x + 5)\)
   c. \((x + 7)(x + 5)\)  
   d. \((x + 7)(x - 5)\)
   e. \((4x + 1)(x + 3)\)  
   f. \((2x - 1)(3x + 2)\)

19. **Reason abstractly.** Examine the products in Item 18. How can you predict the sign of the last term?
Lesson 25-1
Multiplying Binomials

Check Your Understanding

20. Use a graphic organizer to calculate \((6 + 2)^2\). Explain why the product is not \(6^2 + 2^2\).

Determine the product of the binomials using a graphic organizer or by using the Distributive Property.

21. \((x + 7)(x + 2)\)  
22. \((x + 7)(3x - 2)\)

23. Compare the use of the graphic organizer and the use of the Distributive Property to find the product of two binomials.

LESSON 25-1 PRACTICE

Determine each product.

24. \((2 + 1)(3 + 5)\)  
25. \((2 + 3)(2 + 7)\)

26. \((x + 9)(x + 3)\)  
27. \((x + 5)(x + 1)\)

28. \((x - 3)(x + 4)\)  
29. \((x + 1)(x - 5)\)

30. \((x + 3)(x - 3)\)  
31. \((x + 3)(x + 3)\)

32. \((2x - 3)(x - 1)\)  
33. \((x + 7)(3x - 5)\)

34. \((4x + 3)(2x + 1)\)  
35. \((6x - 2)(5x + 1)\)

36. Critique the reasoning of others. A student determined the product \((x - 2)(x - 4)\). Identify and correct the student’s error.

\[
\begin{align*}
(x - 2)(x - 4) &= x(x - 4) - 2(x - 4) \\
&= x^2 - 4x - 2x + 8 \\
&= x^2 - 6x + 8
\end{align*}
\]
Learning Targets:
• Multiply binomials.
• Find special products of binomials.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Look for a Pattern

1. Determine each product.
   a. \((x + 1)(x - 1)\)
   b. \((x + 4)(x - 4)\)
   c. \((x - 7)(x + 7)\)
   d. \((2x - 3)(2x + 3)\)

2. Describe any patterns in the binomials and products in Item 1.

3. Express regularity in repeated reasoning. The product of binomials of the form \((a + b)(a - b)\), has a special pattern called a difference of two squares. Use the patterns you found in Items 1 and 2 to explain how to find the product \((a + b)(a - b)\).
4. Determine each product.
   a. \((x + 1)^2\)  
   b. \((4 + y)^2\)  
   c. \((x + 7)^2\)  
   d. \((2y + 3)^2\)  
   e. \((x - 5)^2\)  
   f. \((4 - x)^2\)  
   g. \((y - 7)^2\)  
   h. \((2x - 3)^2\)  

5. Describe any patterns in the binomials and products in Item 4.

6. **Reason abstractly.** The *square of a binomial*, \((a + b)^2\) or \((a - b)^2\), also has a special pattern. Use the pattern you found in Items 4 and 5 to explain how to determine the square of any binomial.
7. Use the difference of two squares pattern to find the product 
   \((p + k)(p - k)\).
8. Use the square of a binomial pattern to determine \((p + k)^2\).
9. Can you use a special products pattern to determine \((x + 1)(x - 2)\)?
   Explain your reasoning.

**LESSON 25-2 PRACTICE**

Determine each product.
10. \((x - 4)(x + 4)\)
11. \((x + 4)^2\)
12. \((y + 10)(y - 10)\)
13. \((y - 10)^2\)
14. \((2x - 3)^2\)
15. \((2x - 3)(2x + 3)\)
16. \((5x + 1)^2\)
17. \((2y - 1)(2y - 1)\)
18. **Construct viable arguments.** Explain why the products \((x - 3)^2\) and \((x + 3)(x - 3)\) have a different number of terms.
Learning Targets:
- Use a graphic organizer to multiply polynomials.
- Use the Distributive Property to multiply polynomials.

SUGGESTED LEARNING STRATEGIES: Graphic Organizer, Create Representations, Think-Pair-Share, Look for a Pattern

A graphic organizer can be used to multiply polynomials that have more than two terms, such as a binomial times a trinomial. The graphic organizer at right can be used to multiply \((x + 2)(x^2 + 2x + 3)\).

1. Draw a graphic organizer in the space provided in the My Notes section to represent \((x - 3)(x^2 + 5x + 6)\). Label each inner rectangle and find the sum.

\[
\begin{array}{ccc}
  x^2 & x^3 & 2x^2 \\
  2x & 2x^2 & 4x \\
  3x & 6 & \\
\end{array}
\]

\[= x^3 + 4x^2 + 7x + 6\]

2. How many boxes would you need to represent the multiplication of \((x^3 + 5x^2 + 3x - 3)(x^4 - 6x^3 - 7x^2 + 5x + 6)\) using the graphic organizer?

a. Explain how you determined your answer.

b. Use appropriate tools strategically. Would you use the graphic organizer for other multiplications with this many terms? Explain your reasoning.

The Distributive Property can be used to multiply any polynomial by another. Multiply each term in the first polynomial by each term in the second polynomial.

\[(x - 3)(5x^2 - 2x + 1) = 5x^3 - 17x^2 + 7x - 3\]

3. Determine each product.
   a. \(x(x + 5)\)
   b. \((x - 3)(x + 6)\)
   c. \((x + 7)(3x^2 - x - 1)\)
   d. \((3x - 7)(4x^2 + 4x - 3)\)

4. How can you predict the number of terms the product will have before you combine like terms?
5. Are all of the answers to Item 3 polynomials? Justify your response.

6. Explain why the product of two polynomials will always be a polynomial.

7. You can find the product of more than two polynomials, such as 
   
   \((x + 3)(2x + 1)(3x - 2)\).
   
   a. To multiply \((x + 3)(2x + 1)(3x - 2)\), first determine the product of the first two polynomials, \((x + 3)(2x + 1)\).

   \((x + 3)(2x + 1) = \)

   b. Multiply your answer to Part (a) by the third polynomial, \((3x - 2)\).

8. Determine each product.
   
   a. \((x - 2)(x + 1)(2x + 2)\)  
   
   b. \((x + 3)(3x + 1)(2x - 1)\)  
   
   c. \((x - 1)(3x - 2)(x + 4)\)  
   
   d. \((2x - 4)(4x + 1)(3x + 3)\)  

Check Your Understanding

Determine each product.
9. \(a(b + c)\)  
10. \((a + b)(a + c)\)  
11. \((a + b)(a^2 + b + c)\)  
12. \((a + b)(a + c)(b + c)\)

LESSON 25-3 PRACTICE

Determine each product.
13. \(x(x + 7)\)  
14. \(x(2x - 5)\)  
15. \((y + 3)(y + 6)\)  
16. \((y + 3)(y - 6)\)  
17. \(x(2x^2 - 5x + 1)\)  
18. \((x - 1)(2x^2 - 5x + 1)\)  
19. \((2x - 7)(5x^2 - 1)\)  
20. \((2x - 7)(5x^2 - 3x - 1)\)  
21. \((x + 2)(x - 3)(x + 1)\)  
22. \((x + 2)(2x - 3)(2x + 1)\)  

23. Attend to precision. A binomial of degree 2 and variable \(x\) and a trinomial of degree 4 and variable \(x\) are multiplied. What will be the degree of the product? Explain your reasoning.
ACTIVITY 25 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 25-1
Determine each product.

1. \((10 - 3)(10 - 8)\)
2. \((x - 3)(x - 8)\)
3. \((y - 7)(y + 2)\)
4. \((x + 5)(x - 9)\)
5. \((2y - 6)(3y - 8)\)
6. \((4x + 3)(x - 11)\)

7. Which expression represents the area of the rectangle?

- A. \(10x + 9\)
- B. \(9x^2 + 14\)
- C. \(20x + 18\)
- D. \(9x^2 + 25x + 14\)

Lesson 25-2
Determine each product.

8. \((x - 7)(x + 7)\)
9. \((y + 6)(y - 6)\)
10. \((2x - 5)(2x + 5)\)
11. \((3y + 1)(3y - 1)\)
12. \((x - 11)^2\)
13. \((x + 8)^2\)
14. \((2y - 3)^2\)
15. \((3y - 2)^2\)
16. Which expression represents the area of the square?

- A. \(20y^2 + 28\)
- B. \(20y - 28\)
- C. \(25y^2 - 70y + 49\)
- D. \(25y^2 - 49\)
Lesson 25-3
Determine each product.

17. \( x(x^2 - 7) \)
18. \( 2x(x^2 - 3x + 2) \)
19. \( (x + 2)(4x^2 - 7x + 5) \)
20. \( (y - 5)(4y^2 + 5y + 2) \)
21. \( (5x - 9)^2 \)
22. \( (3x - 4)(3x + 4) \)
23. \( (2y + 1)(y^2 + 3y - 5) \)
24. Which expression represents the area of the triangle? Use the formula \( A = \frac{1}{2}bh \).

Determine each product.

25. \( (x - 1)(7x - 1)(x + 2) \)
26. \( (x + 5)(4x - 1)(2x + 3) \)
27. \( (y + 1)^3 \)
28. Devise a plan for finding the product of four polynomials.

MATHEMATICAL PRACTICES
Look for and Make Use of Structure

29. Determine each product and describe any patterns you observe.

\( (x - 1)(x + 1) \)
\( (x - 1)(x^2 + x + 1) \)
\( (x - 1)(x^3 + x^2 + x + 1) \)

From the patterns you see, predict the product of \( (x - 1)(x^4 + x^3 + x^2 + x + 1) \). Describe the pattern that helps you know the answer without needing to multiply.
Employees at Ship-It-Quik must perform computations involving volume and surface area. As part of the job application, potential employees must take a test that involves surface area, volume, and algebraic skills.

1. The surface area of a figure is the total area of all faces. The areas of the faces of a rectangular prism are shown. The surface area of this prism is $18x^2 + 12x + 22$. Complete the first part of the job application by finding the area of the missing face.

The formula for the volume of a rectangular prism is $V = lwh$, where $l$, $w$, and $h$ are length, width, and height, respectively. The formula for the surface area is $SA = 2lw + 2wh + 2lh$.

2. Complete the second part of the job application by verifying whether or not the following computations are correct. Explain your reasoning by showing your work.

   Volume: 
   $2x^2 + 15x + 36$  
   Surface Area: 
   $10x^2 + 90x + 72$

3. Complete the final part of the job application by writing an expression for the volume of a cylinder with radius $3x^2y$ and height $2xy$. Use the formula $V = \pi r^2h$ where $r$ is the radius and $h$ is the height. Simplify your answer as much as possible.
<table>
<thead>
<tr>
<th>Scoring Guide</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mathematics Knowledge and Thinking</strong> (Items 1–3)</td>
<td>• Effective understanding of and accuracy in adding, subtracting, and multiplying polynomials</td>
<td>• Addition, subtraction, and multiplication of polynomials that are usually correct</td>
<td>• Difficulty adding, subtracting, and multiplying polynomials</td>
<td>• Inaccurate addition, subtraction, and multiplication of polynomials</td>
</tr>
<tr>
<td><strong>Problem Solving</strong> (Item 1)</td>
<td>• Appropriate and efficient strategy that results in a correct answer</td>
<td>• Strategy that may include unnecessary steps but results in a correct answer</td>
<td>• Strategy that results in some incorrect answers</td>
<td>• No clear strategy when solving problems</td>
</tr>
<tr>
<td><strong>Mathematical Modeling/Representations</strong> (Items 1–3)</td>
<td>• Clear and accurate understanding of geometric formulas</td>
<td>• Functional understanding of geometric formulas that results in correct answers</td>
<td>• Partial understanding of geometric formulas that results in some incorrect answers</td>
<td>• Little or no understanding of geometric formulas</td>
</tr>
<tr>
<td><strong>Reasoning and Communication</strong> (Item 2)</td>
<td>• Precise use of appropriate math terms and language to justify each step in verifying an answer</td>
<td>• Adequate justification of each step to verify an answer</td>
<td>• Misleading or confusing justification of the steps to verify an answer</td>
<td>• Incomplete or inaccurate justification of the steps to verify an answer</td>
</tr>
</tbody>
</table>
Learning Targets:
- Identify the GCF of the terms in a polynomial.
- Factor the GCF from a polynomial.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Think-Pair-Share, Discussion Groups, Note Taking

Factor Steele Buildings is a company that manufactures prefabricated metal buildings that are customizable. All the buildings come in square or rectangular designs. Most office buildings have an entrance area or great room, large offices, and cubicles. The diagram below shows the front face of one of their designs. The distance \( c \) represents space available for large offices, and \( p \) represents the space available for the great room.

1. To determine how much material is needed to cover the front wall of the building, represent the total area as a product of a monomial and a binomial.

2. Represent the same area from Item 1 as a sum of two monomials.

3. Make use of structure. What property can be used to show that the two quantities in Items 1 and 2 are equal?

4. Factor Steele Buildings inputs the length of the large office space \( c \) into an expression that gives the area of an entire space: \( 6c^2 + 12c - 9 \). Determine the greatest common factor (GCF) of the terms in this polynomial. Explain your choice.
To factor a number or expression means to write the number or expression as a product of its factors.

Example A

<table>
<thead>
<tr>
<th>To Factor a Monomial (the GCF) from a Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Steps to Factoring</strong></td>
</tr>
</tbody>
</table>
| • Determine the GCF of all terms in the polynomial. | $6x^3 + 2x^2 - 8x$  
GCF = $2x$ |
| • Write each term as the product of the GCF and another factor. | $2x(3x^2) + 2x(x) + 2x(-4)$ |
| • Use the Distributive Property to factor out the GCF. | $2x(3x^2 + x - 4)$ |

Try These A

Find the greatest common factor of the terms in each polynomial. Then write each polynomial with the GCF factored out.

a. $36y - 24$

b. $4x^5 - 6x^3 + 10x^2$

c. $15t^2 + 10t - 5$

Check Your Understanding

5. Identify the GCF of the terms in the polynomial $21x^3 + 14x^2 + 35x$.

Factor a monomial (the GCF) from each polynomial.

6. $36x + 9$

7. $6x^4 + 12x^2 - 18x$

8. $125n^6 + 250n^5 + 25n^3$

9. $3x^3 + 9x^2 + 6x$

10. $\frac{2}{3}y^4 + \frac{1}{3}y^3 - \frac{4}{3}y^2$

11. $4x^2y^2 + 12xy^2 - 8x^2y - 4xy$
Lesson 26-1
Factoring by Greatest Common Factor (GCF)

LESSON 26-1 PRACTICE

Use the cylinder for Items 12–15.

The surface area of a cylinder is given by the formula

\[ SA = 2\pi r^2 + 2\pi rh \]

where \( r \) is the radius and \( h \) is the height.

12. Factor a monomial (the GCF) from the formula.

13. Suppose the radius of the cylinder is \( y \) and the height is \( y + 2 \).
   Rewrite the formula in this case, using multiplication and exponent
   rules as needed to simplify the expression.

14. Factor the expression from Item 13 completely.

15. Construct viable arguments. Answer each of the following
   questions and justify your responses.
   a. If the radius of a cylinder doubles, what happens to the GCF of its
      surface area?
   b. What happens to the GCF of the cylinder's surface area if its radius is
      squared?
Learning Targets:
- Factor a perfect square trinomial.
- Factor a difference of two squares.

SUGGESTED LEARNING STRATEGIES: Create Representations, Discussion Groups, Look for a Pattern, Sharing and Responding, Think-Pair-Share

Factor Steele Buildings can create many floor plans with different size spaces. In the diagram below the great room has a length and width of $x$ units, and each cubicle has a length and width of 1 unit. Use the diagram below for Items 1–3.

1. **Model with mathematics.** Represent the area of the entire office above as a sum of the areas of all the rooms.

2. Write the area of the entire office as a product of two binomials.

3. What property can you use to show how the answers to Items 1 and 2 are related? Show this relationship.

4. For each of the following floor plans, write the area of the office as a sum of the areas of all the rooms and as a product of binomials.
   a. 
   b. 
c. What patterns do you observe?

5. Complete the following table. The first row has been done for you.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>1st Factor</th>
<th>2nd Factor</th>
<th>First Term in Each Factor</th>
<th>Second Term in Each Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 6x + 9$</td>
<td>$(x + 3)$</td>
<td>$(x + 3)$</td>
<td>$x$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$(x - 3)$</td>
<td>$(x - 3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(x + 4)$</td>
<td>$(x + 4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(x - 4)$</td>
<td>$(x - 4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(x + 5)$</td>
<td>$(x + 5)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(x - 5)$</td>
<td>$(x - 5)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Express regularity in repeated reasoning. Describe any patterns that you observe in the table from Item 5.

7. Explain how to factor polynomials of the form $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$.

Polynomials of the form $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are called **perfect square trinomials**.

Check Your Understanding

Factor each perfect square trinomial.

8. $x^2 - 14x + 49$  
9. $m^2 + 20m + 100$  
10. $y^2 - 16y + 64$

11. Complete the table by finding the polynomial product of each pair of binomial factors. The first row has been done for you.

<table>
<thead>
<tr>
<th>1st Factor</th>
<th>2nd Factor</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x + 3)$</td>
<td>$(x - 3)$</td>
<td>$x^2 - 9$</td>
</tr>
<tr>
<td>$(x + 4)$</td>
<td>$(x - 4)$</td>
<td></td>
</tr>
<tr>
<td>$(x - 5)$</td>
<td>$(x + 5)$</td>
<td></td>
</tr>
<tr>
<td>$(9 - x)$</td>
<td>$(9 + x)$</td>
<td></td>
</tr>
<tr>
<td>$(2x - 7)$</td>
<td>$(2x + 7)$</td>
<td></td>
</tr>
<tr>
<td>$(6x - 2y)$</td>
<td>$(6x + 2y)$</td>
<td></td>
</tr>
</tbody>
</table>
12. Describe any patterns you observe in the table from Item 11.

13. a. One factor of $36 - y^2$ is $6 + y$. What is the other factor?
   
   b. One factor of $p^2 - 144$ is $p - 12$. What is the other factor?
   
   c. Describe any patterns you observe.

14. Factor each of the following.
   
   a. $49 - x^2$  
   b. $n^2 - 9$  
   c. $64w^2 - 25$
   
   d. Describe any patterns you observe.

15. Explain how to factor a polynomial of the form $a^2 - b^2$.
   
   A polynomial of the form $a^2 - b^2$ is referred to as the **difference of two squares**.

### Check Your Understanding

Factor each difference of two squares.

16. $x^2 - 121$  
17. $16m^2 - 81$  
18. $9 - 25p^2$

### LESSON 26-2 PRACTICE

Identify each polynomial as a perfect square trinomial, a difference of two squares, or neither. Then factor the polynomial if it is a perfect square trinomial or a difference of two squares.

19. $z^2 + 6z + 12$  
20. $4x^2 - 121$  
21. $y^2 - 8y + 16$

22. $y^2 - 8y - 16$  
23. $n^2 + 25$  
24. $169 - 9x^2$

25. What factor would you need to multiply by $(4c + 7)$ to get $16c^2 - 49$?

26. What factor would you need to multiply by $(3d + 1)$ to get $9d^2 + 6d + 1$?

Factor completely. *(Hint: First look for a GCF.)*

27. $2x^2 + 8x + 8$  
28. $3y^2 - 75$  
29. $12x^2 - 12x + 3$

30. **Use appropriate tools strategically.** Explain how you can use your calculator to check that you have factored a polynomial correctly.
ACTIVITY 26 PRACTICE
Write your answers on notebook paper.
Show your work.

Lesson 26-1
1. What is the greatest common factor of the terms in the polynomial $24x^8 + 6x^5 + 9x^2$?
   A. 3   B. $3x^2$   C. 6x   D. $6x^2$

   Factor a monomial (the GCF) from each polynomial.
2. $15x^4 + 20x^3 + 35x$
3. $12m^3 - 8m^2 + 16m + 8$
4. $32y^2 + 48y - 16$
5. $x^3 + x^2 + 3x^3 + 3x^3$

6. Which of these polynomials cannot be factored by factoring out the GCF?
   A. $7x^2 + 14x + 21$   B. $49x^3 + 21x^2 + x$
   C. $x^2 + 14x + 7$   D. $35x^3 + 28x^2 + 7x$

7. The figure shows the dimensions of a garden plot in the shape of a trapezoid. Write and simplify a polynomial for the perimeter of the plot. Then factor the polynomial completely.

   ![Trapezoid Diagram]

8. The area of the rectangle shown below is $6x^2 + 9x$ square feet. The width of the rectangle is given in the figure. What is the length of the rectangle? Justify your answer.

9. Marcus saw the factorization shown below in his textbook, but part of the factorization was covered by a drop of ink. What expression was covered by the drop of ink?

   $-24x^5 - 16x^3 = -8x^3(\blacksquare + 2)$

10. Write a polynomial with four terms that has a GCF of $4x^2$.

Lesson 26-2
Identify each polynomial as a perfect square trinomial, a difference of two squares, or neither. Then factor the polynomial if it is a perfect square trinomial or a difference of two squares.

11. $9x^2 - 121$
12. $m^2 - 16m + 64$
13. $y^2 + 12y - 36$
14. $16z^2 + 25$
15. $25 - 144p^2$
16. $x^2 + 50x + 625$

Factor completely.
17. $2x^2 - 32$
18. $32 - 8p$
19. $3x^3 + 12x^2 + 12x$
20. $4y^3 - 32y^2 + 64y$
21. $5x^4 - 125x^2$

22. What factor would you need to multiply by $(4x - 1)$ to get $16x^2 - 8x + 1$?
   A. $4x - 1$   B. $4x + 1$
   C. $4x^2$   D. $4x$

Use the rectangle for Items 23–25.

![Rectangle Diagram]

23. The area of a rectangle is $64b^2 - 4$ and $W = 8b - 2$. What is $L$?
24. The area of another rectangle is $144c^2 - 4$ and $L = 12c + 2$. What is $W$?
25. Suppose the area of a rectangle is $4x^2 - 4x + 1$ and $L = 2x - 1$.
   
a. What is $W$?
   
b. What must be true about the rectangle in this case? Explain.

26. The area of a square window is given by the expression $m^2 - 16m + 64$. Which expression represents the length of one side of the window?
   
   A. $m - 4$  
   B. $m + 4$  
   C. $m - 8$  
   D. $m + 8$

27. What value of $k$ makes the polynomial $x^2 + 6x + k$ a perfect square trinomial?
   
   A. 3  
   B. 6  
   C. 9  
   D. 36

28. Consider the following values of $c$ in the polynomial $36x^2 + c$.
   
   I. $c = -25$
   
   II. $c = 25$
   
   III. $c = -36$

   Which value or values of $c$ make it possible to factor the polynomial?
   
   A. I only  
   B. I and II only  
   C. I and III only  
   D. I, II, and III

29. Write a perfect square trinomial that includes the term $9x^2$.

30. The polynomial $x^2 + bx + 25$ is a perfect square trinomial. What is the value of $b$? Is there more than one possibility? Explain.

31. Sasha and Pedro were asked to factor the polynomial $9x^2 - 9$ completely and explain their process. Their work is shown below. Has either student factored the polynomial completely? Explain. If not, give the complete factorization.

   **Sasha’s Work**
   
   $9x^2 - 9 = (3x + 3)(3x - 3)$
   
   I used the fact that $9x^2 - 9$ is a difference of two squares.

   **Pedro’s Work**
   
   $9x^2 - 9 = 9(x^2 - 1)$
   
   I factored out the GCF.

32. Which of the following polynomials has $m - 4$ as a factor?
   
   A. $m^2 - 4$  
   B. $m^2 + 16$  
   C. $m^2 - 8m + 16$  
   D. $m^2 - 8m - 16$

33. Given that $x^2 + \square + 100$ is a perfect square trinomial, which of these could be the missing term?
   
   A. 10x  
   B. 20x  
   C. 50x  
   D. 100x

34. Factor $x^4 - 81$ completely. (Hint: Use the fact that $x^4 = (x^2)^2$ to factor $x^4 - 81$ as a difference of two squares. Then consider whether any of the resulting factors can be factored again.)

35. Use the method in Item 34 to factor $y^8 - 625$ completely.

**MATHEMATICAL PRACTICES**

Reason Abstractly and Quantitatively

36. Could a product in the form $(a + b)(a - b)$ ever be equal to $a^2 + b^2$? Justify your answer.
Factoring Trinomials
Deconstructing Floor Plans
Lesson 27-1 Factoring $x^2 + bx + c$

Learning Targets:
• Use algebra tiles to factor trinomials of the form $x^2 + bx + c$.
• Factor trinomials of the form $x^2 + bx + c$.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Think-Pair-Share, Look for a Pattern, Discussion Groups

Recall that Factor Steele Buildings can create many floor plans with different-size spaces. Custom Showrooms has asked Factor Steele Buildings for a floor plan with one great room, five large offices, and six cubicles. Each great room has a length and width equal to $x$ units, each large office has a width of $x$ units and a length of 1 unit, and each cubicle has a length and width of 1 unit.

Factor Steele Buildings proposes the rectangular floor plan shown below.

1. Represent the area of the entire office as a sum of the areas of all the rooms.

2. Write the area of the entire office as a product of two binomials by multiplying the length of the entire office by the width of the entire office.

3. Make use of structure. Multiply the binomials in Item 2 to check that their product is the expression you wrote in Item 1. Justify your steps and name any properties you use to multiply the binomials.
Items 1 through 3 show how to use algebra tiles to factor a trinomial. However, drawing tiles to factor a trinomial can become time-consuming. Analyzing patterns and using graphic organizers can help factor a trinomial of the form \( x^2 + bx + c \) without using tiles.

4. Consider the binomials \((x - 5)\) and \((x + 3)\).
   a. Determine their product.
   
   b. How is the coefficient of the trinomial’s middle term related to the constant terms of the binomials?
   
   c. How is the constant term of the trinomial related to the constant terms of the binomials?

5. Consider the binomials \((x + 6)\) and \((x + 1)\).
   a. Determine their product.
   
   b. How is the coefficient of the trinomial’s middle term related to the constant terms of the binomials?
   
   c. How is the constant term of the trinomial related to the constant terms of the binomials?

6. **Express regularity in repeated reasoning.** Use the patterns you observed in Items 4 and 5 to analyze a trinomial of the form \( x^2 + bx + c \). Describe how the numbers in the binomial factors are related to the constant term \( c \), and to \( b \), the coefficient of \( x \).
Example A
Factor \( x^2 + 12x + 32 \).

Step 1: Create a graphic organizer as shown. Place the first term in the upper left region. Place the last term in the lower right region.

\[
\begin{array}{c|c}
\hline
x^2 & 32 \\
\hline
\end{array}
\]

Step 2: Identify the factors of \( c \) that add to \( b \). Use a table to help you test factors.

<table>
<thead>
<tr>
<th>Factors of 32</th>
<th>Sum of the Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>32 1</td>
<td>32 + 1 = 33</td>
</tr>
<tr>
<td>16 2</td>
<td>16 + 2 = 18</td>
</tr>
<tr>
<td>8 4</td>
<td>8 + 4 = 12</td>
</tr>
</tbody>
</table>

Step 3: Fill in the missing factors and products in the graphic organizer.

\[
\begin{array}{c|c|c}
\hline
x & 8 \\
\hline
x & x^2 & 8x \\
\hline
4 & 4x & 32 \\
\hline
\end{array}
\]

Step 4: Write the original trinomial as the product of two binomials.
\[
x^2 + 12x + 32 = (x + 4)(x + 8)
\]
Try These A

a. Fill in the missing sections of the graphic organizer for the trinomial $x^2 - 6x + 8$. Express the trinomial as a product of two binomials.

   
   \[
   \begin{array}{c|c}
   \hline
   \text{x}^2 & \hline
   \text{-4x} & 8 \\
   \hline
   \end{array}
   \]

b. Make a graphic organizer like the one above for the trinomial $x^2 + 14x + 45$. Express the trinomial as a product of two binomials.

c. Factor $x^2 + 6x - 27$.

d. Factor $x^2 + 10x + 1$.

MATH TIP
If there are no factors of $c$ that add to $b$, the trinomial cannot be factored. A polynomial that cannot be factored is called **unfactorable** or a **prime polynomial**.
Lesson 27-1
Factoring $x^2 + bx + c$

Check Your Understanding

Factor each trinomial. Then multiply your factors to check your work.

7. $x^2 + 15x + 56$
8. $x^2 + 22x + 120$
9. $x^2 + 6x - 27$
10. $x^2 - 14x + 48$
11. $x^2 - x + 1$

LESSON 27-1 PRACTICE

Factor each trinomial.

12. $x^2 + 8x + 15$
13. $x^2 - 5x - 14$
14. $x^2 - 5x + 3$
15. $x^2 - 16x + 48$
16. $24 + 10x + x^2$
17. Custom Showrooms has expanded and now wants Factor Steele Buildings to create a floor plan with one great room, 15 large offices, and 50 cubicles.
   a. Write the area of the new floor plan as a trinomial.
   b. Factor the trinomial.
   c. Multiply the binomials in Part (b) to check your work.
18. Reason abstractly. Suppose $x^2 + bx + c$ is a factorable trinomial in which $c$ is a positive prime number.
   a. Write an expression to represent the value of $b$.
   b. Write $x^2 + bx + c$ as the product of two factors using only $c$ as an unknown constant.

MATH TIP

A prime number has only itself and 1 as factors. For example, the numbers 3 and 11 are prime numbers.
Learning Targets:
• Factor trinomials of the form \( ax^2 + bx + c \) when the GCF is 1.
• Factor trinomials of the form \( ax^2 + bx + c \) when the GCF is not 1.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Note Taking, Guess and Check, Look for a Pattern, Work Backward

Custom Showrooms now wants Factor Steele Buildings to create a floor plan with more than one great room. Instead, Custom Showrooms wants two great rooms, seven large offices, and six cubicles.

The trinomial \( 2x^2 + 7x + 6 \) can be factored to determine the length and width of the entire office space.

1. Attend to precision. How is the trinomial \( 2x^2 + 7x + 6 \) different from the trinomials you factored in Lesson 27-1?

Example A
Factor \( 2x^2 + 7x + 6 \) using a guess and check method.

<table>
<thead>
<tr>
<th>Possible Binomial Factors</th>
<th>Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2x \quad )(x \quad ))</td>
<td>(a = 2) can be factored as (2 \cdot 1).</td>
</tr>
<tr>
<td>((2x \quad + \quad )(x \quad + \quad ))</td>
<td>(c = 6), so both factors have the same sign. (b = 7), so both factors are positive. 6 can be factored as (1 \cdot 6, 6 \cdot 1, 2 \cdot 3, ) or (3 \cdot 2).</td>
</tr>
<tr>
<td>((2x \quad + \quad 1)(x \quad + \quad 6))</td>
<td>Product: (2x^2 + 13x + 6), incorrect</td>
</tr>
<tr>
<td>((2x \quad + \quad 6)(x \quad + \quad 1))</td>
<td>Product: (2x^2 + 8x + 6), incorrect</td>
</tr>
<tr>
<td>((2x \quad + \quad 2)(x \quad + \quad 3))</td>
<td>Product: (2x^2 + 8x + 6), incorrect</td>
</tr>
<tr>
<td>((2x \quad + \quad 3)(x \quad + \quad 2))</td>
<td>Product: (2x^2 + 7x + 6), correct factors</td>
</tr>
</tbody>
</table>

Example B
Factor \( 3x^2 + 8x - 11 \) using a guess and check method.

<table>
<thead>
<tr>
<th>Possible Binomial Factors</th>
<th>Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3x \quad )(x \quad ))</td>
<td>(a = 3) can be factored as (3 \cdot 1).</td>
</tr>
<tr>
<td>((3x \quad + \quad )(x \quad - \quad )) or ((3x \quad - \quad )(x \quad + \quad ))</td>
<td>(c = -11), so the factors have different signs. 11 can be factored as (11 \cdot 1) or (1 \cdot 11).</td>
</tr>
<tr>
<td>((3x \quad + \quad 1)(x \quad - \quad 1))</td>
<td>Product: (3x^2 + 8x - 11), correct factors</td>
</tr>
<tr>
<td>((3x \quad - \quad 11)(x \quad + \quad 1))</td>
<td>Product: (3x^2 - 8x - 11), incorrect</td>
</tr>
<tr>
<td>((3x \quad + \quad 1)(x \quad - \quad 11))</td>
<td>Product: (3x^2 - 32x - 11), incorrect</td>
</tr>
<tr>
<td>((3x \quad - \quad 1)(x \quad + \quad 11))</td>
<td>Product: (3x^2 + 32x - 11), incorrect</td>
</tr>
</tbody>
</table>
Lesson 27-2
Factoring $ax^2 + bx + c$

Try These A–B
Factor the trinomials.

a. $3x^2 + 5x + 2$

b. $2x^2 + 5x - 18$

c. $2x^2 + 6x - 7$

Example C
Factor $4x^2 - 4x - 15$ using a guess and check method.

<table>
<thead>
<tr>
<th>Possible Binomial Factors</th>
<th>Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4x)(x)$, $(2x)(2x)$</td>
<td>$a = 4$ can be factored as $4 \cdot 1$ or $2 \cdot 2$.</td>
</tr>
<tr>
<td>$(4x - 1)(x + 5)$, $(4x + 1)(x - 5)$, $(4x - 15)(x + 1)$, $(4x + 15)(x - 1)$, $(4x - 5)(x + 3)$, $(4x + 5)(x - 5)$, $(4x - 3)(x + 5)$, $(4x + 3)(x - 5)$, $(2x - 1)(2x + 15)$, $(2x + 1)(2x - 15)$, $(2x - 3)(2x + 5)$, $(2x + 3)(2x - 5)$</td>
<td>$c = -15$, so the factors have different signs. $15$ can be factored as $1 \cdot 15$, $15 \cdot 1$, $3 \cdot 5$, or $5 \cdot 3$.</td>
</tr>
</tbody>
</table>

Try These C
Factor the trinomials.

a. $6x^2 - 11x - 2$

b. $6x^2 - 13x - 4$

c. $4x^2 - 20x + 21$
Example D

Factor $9x^2 - 24x + 12$.

Step 1: The coefficients 9, −24, and 12 are all divisible by 3. Factor out the GCF.

$$9x^2 - 24x + 12 = 3(3x^2 - 8x + 4)$$

Step 2: Factor $3x^2 - 8x + 4$ using a guess and check method.

<table>
<thead>
<tr>
<th>Possible Binomial Factors</th>
<th>Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(3x - 4)(x - 1)$</td>
<td>Product: $3x^2 - 7x + 4$, incorrect</td>
</tr>
<tr>
<td>$(3x - 1)(x - 4)$</td>
<td>Product: $3x^2 - 13x + 4$, incorrect</td>
</tr>
<tr>
<td>$(3x - 2)(x - 2)$</td>
<td>Product: $3x^2 - 8x + 4$, correct factors</td>
</tr>
</tbody>
</table>

Solution: Write the complete factorization, including the GCF from Step 1:

$$3(3x - 2)(x - 2)$$

Check: Multiply to check your answer.

$$3(3x - 2)(x - 2) = 3(3)x^2 - 6x - 2x + 4 = 3(3x^2 - 8x + 4) = 9x^2 - 24x + 12$$

Try These D

Factor the trinomials completely. Check your work by multiplying the factors.

a. $10x^2 + 19x + 6$
 b. $8x^2 + 20x - 28$
 c. $8x^3 - 14x^2 + 6x$

Check Your Understanding

Factor each trinomial completely. Check your work by multiplying the factors.

2. $5x^2 + x - 4$
 3. $49x^2 - 126x + 56$
 4. $9x^3 - 39x^2 - 30x$

Lesson 27-2 Practice

Model with mathematics. Factor Steele Buildings has received several floor plan requests. For Items 5–8, factor each floor plan scenario completely to help Factor Steele Buildings determine the space’s dimensions.

5. 3 great rooms, 23 large offices, 14 cubicles
6. 10 great rooms, 31 large offices, 15 cubicles
7. 8 great rooms, 92 large offices, 180 cubicles
8. 12 great rooms, 38 large offices, 20 cubicles
9. Suppose $ax^2 + bx + c$ is a factorable trinomial in which both $a$ and $c$ are positive prime numbers. Write an expression to represent the value of $b$. 
**Activity 27 Practice**

Write your answers on notebook paper. Show your work.

**Lesson 27-1**

Factor each trinomial.

1. \(x^2 + 11x + 30\)
2. \(x^2 + 22x + 121\)
3. \(x^2 + x - 30\)
4. \(x^2 - 7x - 18\)
5. \(x^2 - 169\)
6. \(x^2 + 9x - 36\)

Mrs. Harbrook can choose from two rectangular pool sizes. The pool manufacturer provides her with the area of the pool, but she needs to find the dimensions in order to determine if the pool will fit in her yard. Use the rectangle for Items 7 and 8.

![](image)

7. If the area of the pool is \(x^2 - 17x + 72\), what are possible expressions to represent the length \(L\) and the width \(W\)?

8. a. If the area of the pool is \(x^2 + 24x + 144\), what are possible expressions to represent the length \(L\) and the width \(W\)?
   b. What do these dimensions tell you about the shape of the pool?

The area of a parallelogram is given by the formula \(A = bh\), where \(b\) is the base and \(h\) is the height. Use this information for Items 9 and 10.

![](image)

9. If the area of the parallelogram is \(x^2 + x - 42\), what are possible expressions to represent the base \(b\) and the height \(h\)?

10. If the area of the parallelogram is \(x^2 + 4x - 117\), what are possible expressions to represent the base \(b\) and the height \(h\)?

11. Which of the following trinomials cannot be factored?
   A. \(x^2 + 3x + 2\)  
   B. \(x^2 + 3x - 2\)  
   C. \(x^2 - 3x + 2\)  
   D. \(x^2 + 2x - 3\)

12. Which of the following binomials is a factor of the trinomial \(y^2 - y - 20\)?
   A. \(y - 4\)  
   B. \(y + 4\)  
   C. \(y - 10\)  
   D. \(y + 10\)

For Items 13–15, consider the trinomial \(x^2 + 2x + c\). Determine whether each statement is always, sometimes, or never true.

13. If \(c\) is a prime number, then the trinomial cannot be factored.
14. If \(c\) is an even number, then the GCF of the terms in the trinomial is 2.
15. If \(c < 0\), then the trinomial can be factored.
16. Write a trinomial that can be factored such that one of the binomial factors is \(x - 5\). Explain how you found the trinomial.

**Lesson 27-2**

Factor each trinomial completely.

17. \(3x^2 + 8x - 11\)
18. \(5x^2 - 7x + 2\)
19. \(2x^2 - 9x - 5\)
20. \(3x^2 + 17x - 28\)
21. \(7x^2 + 9x + 2\)
22. \(6x^2 - 11x - 7\)
23. \(12x^2 - 11x + 2\)
24. \(8x^2 + 16x + 6\)
25. Which of the following is not a factor of the trinomial $24x^3 - 6x^2 - 9x$?
   A. $3x$
   B. $4x - 3$
   C. $2x + 1$
   D. $2x - 1$

26. Which binomial is a factor of $4x^2 + 12x + 5$?
   A. $2x + 5$
   B. $2x - 5$
   C. $4x + 1$
   D. $4x - 1$

The volume of a rectangular prism is found using the formula $V = lwh$, where $l$ is the length, $w$ is the width, and $h$ is the height. Use the rectangular prism for Items 27–29.

27. If the volume of a rectangular prism is $6x^3 + 3x^2 - 18x$, what are possible expressions to represent the length, width, and height?

28. If the volume of a rectangular prism is $10x^2 - 55x + 60$, what are possible expressions to represent the length, width, and height?

29. If the volume of a rectangular prism is $12x^2 + 22x + 6$, what are possible expressions to represent the length, width, and height?

30. For which value of $k$ is it possible to factor the trinomial $2x^2 + 3x + k$?
   A. $-1$
   B. $1$
   C. $2$
   D. $3$

31. Which of the following trinomials has the binomial $x + 1$ as a factor?
   A. $2x^2 - x - 1$
   B. $2x^2 - 3x + 1$
   C. $3x^2 - 5x + 2$
   D. $3x^2 + x - 2$

32. Mayumi was asked to completely factor the trinomial $4x^2 + 10x + 4$. Her work is shown below. Is her solution correct? Justify your response.

4 can be factored as $4 \cdot 1$ or $2 \cdot 2$.
Try $(2x + ) (2x + )$.
$(2x + 2)(2x + 2) = 4x^2 + 8x + 4$; incorrect
$(2x + 4)(2x + 1) = 4x^2 + 10x + 4$; correct!
The factorization is $(2x + 4)(2x + 1)$.

33. Given that the trinomial $5x^2 + bx + 10$ can be factored, which of the following statements must be true?
   A. The value of $b$ must be positive.
   B. The value of $b$ must be negative.
   C. The value of $b$ cannot be 3.
   D. The value of $b$ cannot be $-7$.

34. What is the factorization of the trinomial $p^2x^2 - 2pqx + q^2$?

35. Write a trinomial of the form $ax^2 + bx + c$ (with $a = 1$) that cannot be factored into binomial factors. Explain how you know the trinomial cannot be factored.

36. The area of a rectangular carpet is $6x^2 - 11x + 4$ square yards. The length of the carpet is $3x - 4$ yards. Which of the following is the width?
   A. $2x - 1$ yards
   B. $2x + 1$ yards
   C. $3x - 1$ yards
   D. $3x + 1$ yards

**MATHEMATICAL PRACTICES**

**Construct Viable Arguments and Critique the Reasoning of Others**

37. Guillaume is asked to factor a trinomial of the form $x^2 + bx - 8$. He says that because the constant term is negative, both binomial factors of the trinomial will involve subtraction. Is he correct? Explain.
A field trip costs $800 for the charter bus plus $10 per student for \( x \) students. The cost per student is represented by the expression \( \frac{10x + 800}{x} \).

The cost-per-student expression is a rational expression. A **rational expression** is a ratio of two polynomials.

Like fractions, rational expressions can be simplified and combined using the operations of addition, subtraction, multiplication, and division.

When a rational expression has a polynomial in the numerator and a monomial in the denominator, it may be possible to simplify the expression by dividing each term of the polynomial by the monomial.

### Example A
Simplify by dividing: \( \frac{12x^5 + 6x^4 - 9x^3}{3x^2} \)

**Step 1:** Rewrite the rational expression to indicate each term of the numerator divided by the denominator.

\[
\frac{12x^5}{3x^2} + \frac{6x^4}{3x^2} - \frac{9x^3}{3x^2}
\]

**Step 2:** Divide. Use the Quotient of Powers Property.

\[
\frac{4x^3}{3} + \frac{2x^2}{3} - \frac{3x^1}{3}
\]

**Solution:** \( 4x^3 + 2x^2 - 3x \)

### Try These A
Simplify by dividing.

a. \( \frac{5y^4 - 10y^3 - 5y^2}{5y^2} \)

b. \( \frac{32n^6 - 24n^4 + 16n^2}{-8n^2} \)
Lesson 28-1
Simplifying Rational Expressions

To simplify a rational expression, first factor the numerator and denominator. Remember that factors can be monomials, binomials, or even polynomials. Then, divide out the common factors.

Example B
Simplify \( \frac{12x^2}{6x^3} \).

Step 1: Factor the numerator and denominator.
\[ \frac{2 \cdot 6 \cdot x \cdot x}{6 \cdot x \cdot x \cdot x} \]

Step 2: Divide out the common factors.
\[ \frac{2 \cdot \cancel{6} \cdot x \cdot x}{\cancel{6} \cdot x \cdot x \cdot x} \]

Solution: \( \frac{2}{x} \)

Example C
Simplify \( \frac{2x^2 - 8}{x^2 - 2x - 8} \).

Step 1: Factor the numerator and denominator.
\[ \frac{2(x + 2)(x - 2)}{(x + 2)(x - 4)} \]

Step 2: Divide out the common factors.
\[ \frac{2(x + 2)(x - 2)}{(x + 2)(x - 4)} \]

Solution: \( \frac{2(x - 2)}{x - 4} \)

Try These B–C
Simplify each rational expression.

a. \( \frac{6x^4 y}{15xy^3} \)

b. \( \frac{x^2 + 3x - 4}{x^2 - 16} \)

c. \( \frac{15x^2 - 3x}{25x^2 - 1} \)

The value of the denominator in a rational expression cannot be zero because division by zero is undefined.

- In Example B, \( x \) cannot equal 0 because \( 6 \cdot (0)^3 = 0 \).
- To find the excluded values of \( x \) in Example C, first factor the denominator. This shows that \( x = -2 \) because that would make the factor \( x + 2 = 0 \). Also, \( x \neq 4 \) because that would make the factor \( x - 4 = 0 \). Therefore, in Example C, \( x \) cannot equal \(-2 \) or \( 4 \).
Lesson 28-1
Simplifying Rational Expressions

Example D
Divide \( \frac{1-x}{x-1} \). Simplify your answer if possible.

Step 1: Factor the numerator.
\[-1(x-1)\]

Step 2: Divide out the common factor.
\[\frac{-1(x-1)}{x-1}\]

Solution: \(-1\)

Try These D
Divide. Simplify your answer if possible. Identify any excluded values of the variable.

a. \( \frac{x-5}{5-x} \)

b. \( \frac{3x-3}{1-x} \)

Check Your Understanding
Attend to precision. Describe the steps you would take to simplify each rational expression. Identify any excluded values of the variable.

1. \( \frac{x^2 - 36}{6-x} \)

2. \( \frac{x^2 - 10x + 24}{4x - 16} \)

LESSON 28-1 PRACTICE
Simplify by dividing.

3. \( \frac{16x^3 - 8x^3 + 4x^2}{4x^2} \)

4. \( \frac{15x^6 - 20x^4}{-5x^3} \)

5. \( \frac{24x^6 + 18x^5 - 15x^3 + 12x^2}{3x^2} \)

Simplify.
6. \( \frac{3x^2yz}{12xyz^3} \)

7. \( \frac{25x^4y^3z^4}{-5x^5y^3z^3} \)

8. \( \frac{x^2 - 2x + 1}{x^2 + 3x - 4} \)

9. \( \frac{2x^2}{4x^3 - 16x} \)

10. \( \frac{x + 1}{-4 - 4x} \)

11. \( \frac{x^2 + 6x + 9}{x^2 - 9} \)

12. Model with mathematics. The four algebra classes at Sanchez School are going on a field trip to a museum. Each class contains \( s \) students. The museum charges $8 per student for admission. There is also a flat fee of $200 for the buses.

a. Write an expression for the total cost of the buses and the museum admission fees for all four classes.

b. Write a rational expression for the cost per student. Simplify the expression as much as possible.

c. Use the expression you wrote in Part (b) to find the cost per student if each class has 20 students.
Learning Targets:
- Divide a polynomial of degree one or two by a polynomial of degree one or two.
- Express the remainder of polynomial division as a rational expression.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Identify a Subtask, Close Reading, Note Taking, Discussion Groups

Division of polynomials is similar to long division of real numbers.

Example A
Divide \( \frac{525}{25} \) using long division.

Step 1: Divide 52 by 25.

\[
\begin{array}{c|c}
25 & 525 \\
25 & -50 \\
\hline
\end{array}
\]

Step 2: Bring down 5.

\[
\begin{array}{c|c}
25 & 525 \\
25 & -50 \\
\hline
\end{array}
\]

Step 3: Divide 25 by 25.

\[
\begin{array}{c|c}
21 & 525 \\
25 & -50 \\
25 & -25 \\
\hline
0 & 0 \\
\end{array}
\]

Solution: The quotient is 21.
Division with polynomials can be done in the same way as long division with whole numbers.

**Example B**

Simplify using long division: \( \frac{12x^5 + 6x^4 - 9x^3}{3x^2} \)

**Step 1:** Divide \( 12x^5 \) by \( 3x^2 \).

\[
\begin{array}{c|ccccc}
 & 4x^3 & \downarrow & & & \\
\hline
3x^2 & 12x^5 & + & 6x^4 & - & 9x^3 \\
\downarrow & 12x^5 & & & & \\
0 & & & & & \\
\end{array}
\]

**Step 2:** Bring down \( 6x^4 \).

\[
\begin{array}{c|ccccc}
 & 4x^3 & 6x^4 & \downarrow & & \\
\hline
3x^2 & 12x^5 & + & 6x^4 & - & 9x^3 \\
\downarrow & 12x^5 & & & & \\
0 & & & & & \\
\end{array}
\]

**Step 3:** Divide \( 6x^4 \) by \( 3x^2 \).

\[
\begin{array}{c|ccccc}
 & 4x^3 + 2x^2 & \downarrow & & & \\
\hline
3x^2 & 12x^5 & + & 6x^4 & - & 9x^3 \\
\downarrow & 12x^5 & & & & \\
6x^4 & & & & & \\
\downarrow & 6x^4 & & & & \\
0 & & & & & \\
\end{array}
\]

**Step 4:** Bring down \( -9x^3 \).

\[
\begin{array}{c|ccccc}
 & 4x^3 + 2x^2 & -9x^3 & \downarrow & & \\
\hline
3x^2 & 12x^5 & + & 6x^4 & - & 9x^3 \\
\downarrow & 12x^5 & & & & \\
6x^4 & & & & & \\
\downarrow & 6x^4 & & & & \\
-9x^3 & & & & & \\
\end{array}
\]

**Step 5:** Divide \( -9x^3 \) by \( 3x^2 \).

\[
\begin{array}{c|ccccc}
 & 4x^3 + 2x^2 - 3x & \downarrow & & & \\
\hline
3x^2 & 12x^5 & + & 6x^4 & - & 9x^3 \\
\downarrow & 12x^5 & & & & \\
6x^4 & & & & & \\
\downarrow & 6x^4 & & & & \\
-9x^3 & & & & & \\
\downarrow & -9x^3 & & & & \\
0 & & & & & \\
\end{array}
\]

**Solution:** The quotient is \( 4x^3 + 2x^2 - 3x \).
Example C

Simplify using long division: \( \frac{x^2 + 9x + 14}{x + 7} \).

**Step 1:** Divide \( x^2 \) by \( x \).

\[
\begin{array}{c|cc}
 & x & \\
\hline
x + 7 & x^2 + 9x + 14 \\
 & - (x^2 + 7x) \\
\end{array}
\]

**Step 2:** Distribute the negative and subtract \( x^2 + 7x \) from \( x^2 + 9x \).

\[
\begin{array}{c|cc}
 & x \\
\hline
x + 7 & x^2 + 9x + 14 \\
 & - x^2 - 7x \\
 & \hline
 & 2x + 14
\end{array}
\]

**Step 3:** Bring down the next term, 14.

\[
\begin{array}{c|cc}
 & x + 2 \\
\hline
x + 7 & x^2 + 9x + 14 \\
 & - x^2 - 7x \\
 & \hline
 & 2x + 14
\end{array}
\]

**Step 4:** Divide \( 2x \) by \( x \).

\[
\begin{array}{c|cc}
 & x + 2 \\
\hline
x + 7 & x^2 + 9x + 14 \\
 & - x^2 - 7x \\
 & \hline
 & 2x + 14
\end{array}
\]

**Step 5:** Distribute the negative and subtract \( 2x + 14 \) from \( 2x + 14 \).

\[
\begin{array}{c|cc}
 & x + 2 \\
\hline
x + 7 & x^2 + 9x + 14 \\
 & - x^2 - 7x \\
 & \hline
 & 2x + 14 \\
 & \hline
 & 0
\end{array}
\]

**Solution:** The quotient is \( x + 2 \).

**Try These A–B–C**

Simplify using long division.

a. \( \frac{24x^5 - 8x^4 + 12x^3 - 4x^2}{4x^2} \)

b. \( \frac{x^2 - x - 12}{x - 4} \)
Lesson 28-2
Dividing Polynomials

Sometimes there are remainders when dividing integers. In a similar way, sometimes there are remainders when dividing polynomials.

Example D
Simplify using long division: \( \frac{2x^3 - 6x + 15}{x + 1} \).

**Step 1:** Divide. Add the term \( 0x^2 \) to the dividend as a placeholder.

\[
\begin{array}{c|ccccc}
& 2x^2 & -2x & -4 \\
\hline
x + 1 & 2x^3 & +0x^2 & -6x & +15 \\
\hline
& - (2x^3 + 2x^2) \\
& & -2x^2 & -6x \\
& & - (-2x^2 - 2x) \\
& & & -4x & +15 \\
& & & - (-4x - 4) \\
& & & & 19 \\
\end{array}
\]

**Step 2:** Write the remainder as \( \frac{19}{x + 1} \).

\[
\begin{array}{c|ccccc}
& 2x^2 & -2x & -4 & \frac{19}{x + 1} \\
\hline
x + 1 & 2x^3 & +0x^2 & -6x & +15 \\
\hline
& - (2x^3 + 2x^2) \\
& & -2x^2 & -6x \\
& & - (-2x^2 - 2x) \\
& & & -4x & +15 \\
& & & - (-4x - 4) \\
& & & & 19 \\
\end{array}
\]

**Solution:** The quotient is \( 2x^2 - 2x - 4 + \frac{19}{x + 1} \).

**Try These D**
Simplify using long division.

a. \( (3x^2 + 6x + 1) \div (3x) \)

b. \( (6x^2 + 5x - 20) \div (3x + 4) \)

c. \( (3x^3 + 5x - 10) \div (x - 4) \)
1. **Make sense of problems.** Consider the following polynomial division problem: \[ \frac{3x^2 - 8x + 15}{x^2 + 3x - 4} \]

   a. How does this division problem differ from those in the examples?

   b. Use long division to perform the division.

   c. Write the remainder as a rational expression.

   d. What is the quotient?

2. **Make use of structure.** Describe how dividing polynomials using long division is similar to dividing whole numbers using long division.

3. Explain how to check a division problem involving whole numbers that has a remainder.

4. Explain how to check a division problem involving polynomials that has a remainder. To demonstrate, use \((3x^3 + x - 2) \div (x^3 + 2x + 3) = 3 + \frac{-5x - 11}{x^3 + 2x + 3}\).

### Check Your Understanding

5. Simplify using long division.
   - \[ \frac{4x^2 + 6x}{2x} \]
   - \[ \frac{12x^5 + 24x^4 - 16x^3 - 12x^2}{-4x^2} \]
   - \[ \frac{5x^2 - 21x + 4}{5x - 1} \]
   - \[ \frac{3x^2 + 6x - 9}{x + 1} \]

6. \[ \frac{3x^4 - 9x^3 + 6x^2}{3x^2} \]

7. \[ \frac{3x^3 - 6x - 24}{3x - 6} \]

8. \[ \frac{12x^2 - 15}{x + 5} \]

9. \[ \frac{25x^2 + 20x - 15}{5x^2 + 5x + 5} \]

13. **Reason abstractly.** The area of a rectangular swimming pool is \(2x^2 + 11x + 4\) square feet. The width of the pool is \(x - 2\) feet.

   a. Write a rational expression that represents the length of the pool.

   b. What are the length, width, and area of the pool when \(x = 19\)?
Lesson 28-3
Multiplying and Dividing Rational Expressions

Learning Targets:
• Multiply rational expressions.
• Divide rational expressions.

SUGGESTED LEARNING STRATEGIES: Note Taking, Close Reading

To multiply rational expressions, first factor the numerator and denominator of each expression. Next, divide out any common factors. Then simplify, if possible.

Example A
Multiply $\frac{2x-4}{x^2-1} \cdot \frac{3x+3}{x^2-2x}$. Simplify your answer if possible.

Step 1: Factor the numerators and denominators.
$$\frac{2(x-2)}{(x+1)(x-1)} \cdot \frac{3(x+1)}{x(x-2)}$$

Step 2: Divide out common factors.
$$\frac{2(x-2) \cdot 3(x+1)}{(x+1)(x-1)(x)(x-2)}$$

Solution: $\frac{6}{x(x-1)}$

Try These A
Multiply. Simplify your answer.

a. $\frac{y^2 + 5y + 6}{y + 2} \cdot \frac{y}{2y + 6}$   b. $\frac{2x + 2}{x^2 - 16} \cdot \frac{x^2 - 5x + 4}{4x^2 - 4}$

To divide rational expressions, use the same process as dividing fractions. Write the division as multiplication of the reciprocal. Then simplify.

Example B
Divide: $\frac{x^2 - 5x + 6}{x^2 - 9} \div \frac{2x - 4}{x^2 + 2x - 3}$. Simplify your answer.

Step 1: Rewrite the division as multiplication by the reciprocal.
$$\frac{x^2 - 5x + 6}{x^2 - 9} \cdot \frac{x^2 + 2x - 3}{2x - 4}$$

Step 2: Factor the numerators and the denominators.
$$\frac{(x-2)(x-3)}{(x+3)(x-3)} \cdot \frac{(x+3)(x-1)}{2(x-2)}$$

Step 3: Divide out common factors.
$$\frac{(x-2)(x-3)(x+3)(x-1)}{(x+3)(x-3)(2)(x-2)}$$

Solution: $\frac{x-1}{2}$

MATH TIP
When dividing fractions, write the division as multiplication by the reciprocal.
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

If $a$, $b$, $c$, and $d$ have any common factors, you can divide them out before you multiply.

$$\frac{4}{15} \div \frac{8}{3} = \frac{4}{15} \cdot \frac{3}{8} = \frac{1}{10}$$
Lesson 28-3
Multiplying and Dividing Rational Expressions

Try These B
Divide. Simplify your answer.

a. \( \frac{w^2 - 2w - 3}{w^2 - 6w + 9} \div \frac{5}{w - 3} \)

b. \( \frac{3xy}{3x^2 - 12} \div \frac{xy + y}{x^2 + 3x + 2} \)

Check Your Understanding

1. Critique the reasoning of others. A student was asked to divide the rational expressions shown below. Examine the student’s solution, and then identify and correct the error.

\[
\frac{a^2 - 9}{3a} \div \frac{a + 3}{a - 3} = \frac{3a}{a^2 - 9} \cdot \frac{a + 3}{a - 3}
\]

\[
= \frac{3a}{(a + 3)(a - 3)} = \frac{3a}{a^2 - 9}
\]

2. What is the quotient when \( \frac{2x + 6}{x + 5} \) is divided by \( \frac{2}{x + 5} \)?

LESSON 28-3 PRACTICE
Multiply or divide.

3. \( \frac{x^2 - 5x - 6}{x^2 - 4} \cdot \frac{x + 2}{x^2 - 12x + 36} \)

4. \( \frac{x^3}{x^2 - 1} \cdot \frac{2x + 2}{4x} \)

5. \( \frac{x^2 - y^2}{12} \div \frac{36}{x + y} \)

6. \( (b^2 + 12b + 11) \cdot \frac{b + 9}{b^2 + 20b + 99} \)

7. \( \frac{x^2 + 4x + 4}{2x + 4} \div \frac{x^2 - 4}{x^2 - 6x + 5} \)

8. \( \frac{1}{x - 1} \div \frac{x}{x - 1} \)

9. \( \frac{2x + 4}{x^2 + 11x + 18} \div \frac{x + 1}{x^2 + 14x + 45} \)

10. \( \frac{m^2 + m - 6}{m^2 + 8m + 15} \div \frac{m^3 - m - 2}{m^2 + 9m + 20} \)

11. Make sense of problems. The figure shows a rectangular prism.

The area of the rectangular face \( ABCD \) is \( x^2 + 2x - 15 \).

a. The length of edge \( \overline{DC} \) is \( x + 1 \). Write a rational expression that represents the length of edge \( \overline{BC} \).

b. The length of edge \( \overline{BF} \) is \( \frac{x^2 + 2x + 1}{x + 5} \). Write and simplify a product to find the area of face \( BFGC \).
Lesson 28-4
Adding and Subtracting Rational Expressions

Learning Targets:
- Identify the least common multiple (LCM) of algebraic expressions.
- Add and subtract rational expressions.

SUGGESTED LEARNING STRATEGIES: Note Taking, Close Reading, Sharing and Responding, Identify a Subtask

To add or subtract rational expressions with the same denominator, add or subtract the numerators and then simplify if possible.

Example A
Simplify $\frac{10}{x} - \frac{5}{x}$.

Step 1: Subtract the numerators.

Solution: $\frac{5}{x}$

Example B
Simplify $\frac{2x}{x+1} + \frac{2}{x+1}$.

Step 1: Add the numerators.

Step 2: Factor.

Step 3: Divide out common factors.

Solution: 2

Try These A–B
Add or subtract. Simplify your answer.

a. $\frac{3}{x^2} - \frac{x}{x^2}$

b. $\frac{2}{x+3} - \frac{6}{x+3} + \frac{x}{x+3}$

c. $\frac{x}{x^2 - x} + \frac{4x}{x^2 - x}$

To add or subtract rational expressions with unlike denominators, first identify a common denominator. The least common multiple (LCM) of the denominators is used for the common denominator.

One way to determine the LCM is to factor each expression. The LCM is the product of each factor common to the expressions as well as any non-common factors.

MATH TERMS
The least common multiple is the smallest multiple that two or more numbers or expressions have in common.

The numbers 10 and 25 have many common multiples. The number 50 is the least common multiple.
Example C

Determine the LCM of \(x^2 - 4\) and \(2x + 4\).

**Step 1:** Factor each expression.

\[x^2 - 4 = (x + 2)(x - 2)\]
\[2x + 4 = 2(x + 2)\]

**Step 2:** Identify the factors.

Common Factor: \((x + 2)\)

Factors Not in Common: \(2\) and \((x - 2)\)

**Step 3:** The LCM is the product of all of the factors in Step 2.

**Solution:** The LCM is \(2(x + 2)(x - 2)\).

Try These C

a. Determine the LCM of \(2x + 2\) and \(x^2 + x\). Use the steps below.

Factor each expression:

Common Factor(s):

Factors Not in Common:

LCM:

b. Determine the LCM of \(x^2 - 2x - 15\) and \(3x + 9\).

Now add and subtract rational expressions with different denominators. First, determine the LCM of the denominators. Next, write each fraction with the LCM as the denominator. Then, add or subtract. Simplify if possible.

Example D

Subtract \(\frac{2}{x} - \frac{3}{x^2 - 2x}\). Simplify your answer if possible.

**Step 1:** Determine the LCM.

Factor the denominators: \(x\) and \(x(x - 2)\)

The LCM is \(x(x - 2)\).

**Step 2:** Multiply the numerator and denominator of the first term by \((x - 2)\). The denominator of the second term is the LCM.

\[\frac{2}{x} \cdot \frac{(x - 2)}{(x - 2)} = \frac{3}{x(x - 2)}\]

**Step 3:** Use the Distributive Property in the numerator.

\[\frac{2x - 4}{x(x - 2)} - \frac{3}{x(x - 2)}\]

**Step 4:** Subtract the numerators.

\[\frac{2x - 7}{x(x - 2)}\]

**Solution:** \(\frac{2x - 7}{x(x - 2)}\)
Lesson 28-4
Adding and Subtracting Rational Expressions

Example E
Add \(-\frac{4}{5-p} + \frac{3}{p-5}\). Simplify your answer if possible.

Step 1: Determine a common denominator. \(p - 5\)

Step 2: Multiply the numerator and denominator of the first term by \(-1\).

\[
\frac{-4}{5-p} \cdot \frac{-1}{-1} + \frac{3}{p-5}
\]

Step 3: Multiply.

\[
\frac{4}{p-5} + \frac{3}{p-5}
\]

Step 4: Add.

\[
\frac{7}{p-5}
\]

Solution: \(\frac{7}{p-5}\)

Try These D–E

a. Add \(\frac{1}{x^2-1} + \frac{2}{x+1}\). Use the steps below.

Factor each denominator:
Common Factors:
Factors Not in Common:
LCM:
Factor the denominator of the first term. Multiply the numerator and denominator of the second term by \__________________________:

Add the numerators:

Use the Distributive Property:

Combine like terms:

Solution:

Add or subtract. Simplify your answer.

b. \(\frac{3}{x+1} - \frac{x}{x-1}\)

c. \(\frac{2}{x} - \frac{3}{x^2-3x}\)

d. \(\frac{2}{x^2-4} + \frac{x}{x^2+4x+4}\)

MATH TIP
If you multiply (5 - p) by -1, the product is p - 5.
Lesson 28-4

Check Your Understanding

1. Make use of structure. Sometimes the denominator of one fraction or one rational expression works as a common denominator for all fractions or rational expressions in a set.
   a. Write two fractions (rational numbers) in which the denominator of one of the fractions is a common denominator.
   b. Write two rational expressions in which the denominator of one of the expressions is a common denominator.
   c. Show how to add the two rational expressions you wrote in Part (b).

2. List the steps you usually use to add or subtract rational expressions with unlike denominators.

LESSON 28-4 PRACTICE

Determine the least common multiple of each set of expressions.

3. $2x + 4$ and $x^2 - 4$
4. $2x - 8$ and $x - 4$
5. $x - 3$ and $x + 3$
6. $x + 6, x + 7, \text{ and } x^2 + 7x + 6$
7. $x + 3, x^2 + 6x + 9, \text{ and } x^2 - 7x - 30$

Perform the indicated operation.

8. \(\frac{x}{x + 1} - \frac{2}{x + 3}\)
9. \(\frac{2}{3x - 3} - \frac{x}{x^2 - 1}\)
10. \(\frac{x}{x + 5} - \frac{2}{x + 3}\)
11. \(\frac{3}{x - 3} - \frac{x}{x + 4}\)
12. \(\frac{x}{3x - 2} + \frac{2}{x - 5}\)
13. \(\frac{x - 2}{x^2 + 4x + 4} + \frac{x - 2}{x + 2}\)

14. Model with mathematics. In the past week, Emilio jogged for a total of 7 miles and biked for a total of 7 miles. He biked at a rate that was twice as fast as his jogging rate.
   a. Suppose Emilio jogs at a rate of \(r\) miles per hour. Write an expression that represents the amount of time he jogged last week and an expression that represents the amount of time he biked last week. \((\text{Hint: distance} = \text{rate} \times \text{time}, \text{so time} = \frac{\text{distance}}{\text{rate}})\)
   b. Write and simplify an expression for the total amount of time Emilio jogged and biked last week.
   c. Emilio jogged at a rate of 5 miles per hour. What was the total amount of time Emilio jogged and biked last week?
Simplifying Rational Expressions
Totally Rational

ACTIVITY 28 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 28-1
1. Allison correctly simplified the rational expression shown below by dividing.

\[
\frac{35x^7 + 15x^5 - 10x^3}{5x^3}
\]

Which of these is a term in the resulting expression?
A. \(3x^8\)  
B. \(3x^4\)  
C. \(-2x\)  
D. \(-2\)

For Items 2–5, simplify each expression.

2. \(\frac{56x^2y}{70x^3y}\)

3. \(\frac{28x^2}{49xy}\)

4. \(\frac{x^2 - 25}{5x + 25}\)

5. \(\frac{x + 5}{x^2 + x - 20}\)

6. Which of the following expressions is equivalent to a negative integer?
A. \(\frac{5y + 5}{5y - 5}\)  
B. \(\frac{6y - 6}{3 - 3y}\)  
C. \(\frac{2 - 2y}{4y - 4}\)  
D. \(\frac{8y - 8}{4y - 4}\)

7. A rental car costs $24 plus $3 per mile.
   a. Write an expression that represents the total cost of the rental if you drive the car \(m\) miles.
   b. Write and simplify an expression that represents the cost per day if you keep the car for 3 days.
   c. What is the cost per day if you drive 50 miles?

8. The expression \(\frac{x^2 + 8x + c}{x + 4}\) can be simplified to \(x + 4\). What is the value of \(c\)?
   A. \(-16\)  
   B. \(0\)  
   C. \(16\)  
   D. \(64\)

Lesson 28-2
For Items 9–14, determine each quotient by using long division.

9. \((3x^2 + 6x + 2) ÷ 3x\)

10. \((3x^2 - 7x - 6) ÷ (3x + 2)\)

11. \(\frac{2x^2 - 7x - 16}{2x + 3}\)

12. \(\frac{x^2 - 19x + 9}{x - 4}\)

13. \(\frac{4x^2 + 17x - 1}{4x + 1}\)

14. \(\frac{5x^3 + x - 2}{x - 1}\)

15. The area \(A\) and length \(l\) of a rectangle are shown below. Write a rational expression that represents the width \(w\) of the rectangle. Then simplify the expression using long division.

\[
A = (x^2 + 4x + 9) \text{ cm}^2
\]

\[l = (x + 4) \text{ cm}\]

16. Greg was asked to simplify each expression below using long division. For which expression should he have a remainder?
   A. \(\frac{6x^2 + 9x + 3}{3x}\)  
   B. \(\frac{8x^4 + 12x^3 + 16x^2}{4x^2}\)  
   C. \(\frac{15x^2 + 5x + 25}{5}\)  
   D. \(\frac{6x^4 + 12x^3 + 6x^2}{6x}\)

17. A student performed the long division shown below. Is the student's work correct? Justify your response.

\[
\frac{4x^2 + 6x - 11}{x^2 + 2x - 5}
\]

The quotient is \(4 + \frac{2x - 9}{x^2 + 2x - 5}\).
Lesson 28-3
Multiply or divide. Simplify your answer if possible.

18. \( \frac{x + 4}{3x} \cdot \frac{4x^2}{x^2 + 9x + 20} \)

19. \( \frac{3x + 9}{x} \cdot \frac{x^2}{x^2 - 9} \)

20. \( \frac{x^2 - x - 6}{x^2 - 9} \cdot \frac{x^2 + 7x + 12}{x^2 + 4x + 4} \)

21. \( \frac{x^2 - 25}{x^2 - 10x + 25} \div \frac{x^2 + 10x + 25}{2x - 10} \)

22. \( \frac{n^2 - 4n - 5}{n^2 + 2n + 1} \div \frac{n^2 - 6n + 5}{n^2 - 1} \)

In the expression \( \frac{1}{(x + 5)^2} \div \frac{k}{(x + 5)}, k \) is a real number with \( k \neq 0 \). For Items 23–25, determine whether each statement is always, sometimes, or never true.

23. The expression may be simplified so that the variable \( x \) does not appear.

24. The value of the expression is a real number less than 1.

25. When \( k > 0 \), the value of the expression is also greater than 0.

26. A student was asked to divide the rational expressions shown below. Examine the student’s solution, then identify and correct the error.

\[
\frac{x^2 - 6x + 9}{5x} \div \frac{x - 3}{x + 3} = \frac{(x + 3)(x - 3)}{5x} \cdot \frac{x + 3}{x - 3} = \frac{(x + 3)^2}{5x}
\]

27. Which expression is equivalent to \( \frac{3x + 3}{x^2} \cdot \frac{x^2 - x}{x^2 - 1} \)?

A. \( 3 \)

B. \( \frac{3}{x} \)

C. \( 3 - x \)

D. \( \frac{3x + 3}{x} \)

Lesson 28-4
For Items 28–31, determine the least common multiple of each set of expressions.

28. \( x^2 - 25 \) and \( x + 5 \)

29. \( y + 3, y, \) and \( y^2 \)

30. \( x^2 + 5x + 6 \) and \( x^2 + 7x + 12 \)

31. \( x^2 - 4x + 4, x - 2, \) and \( (x - 2)^3 \)

32. Which pair of expressions has a least common multiple that is the product of the expressions?

A. \( x + 7 \) and \( x^2 + 14x + 49 \)

B. \( x + 7 \) and \( x - 7 \)

C. \( x - 3 \) and \( x^2 - 9 \)

D. \( x - 3 \) and \( (x - 3)^2 \)

Add or subtract. Express in simplest form.

33. \( \frac{4}{x} + \frac{3}{x} \)

34. \( \frac{x}{2} + \frac{x}{2} \)

35. \( \frac{x}{x + 1} + \frac{1}{x + 1} \)

36. \( \frac{x}{x^2 - 4x} - \frac{5x}{x - 4} \)

37. \( \frac{3x + 2}{3x - 6} + \frac{x + 2}{x^2 - 4} \)

38. \( -\frac{18}{3 - x} + \frac{7}{x - 3} \)

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

39. Justine lives one mile from the grocery store.
While she was driving to the store, there was a lot of traffic. On her way home, there was no traffic at all, and her average rate (speed) was twice the average rate of her trip to the store.

a. Let \( r \) represent Justine’s average rate on her way to the store. Write an expression for the time it took her to get to the store. (Hint: distance = rate \times time, so time = \frac{distance}{rate}.)

b. Write an expression for the time it took Justine to drive home.

c. Write and simplify an expression for the total time of the round trip to and from the store.

d. If Justine drove at 30 miles per hour to the store, what was the total time for the round trip? Write your answer in minutes.
Rockstar Platforms sets up outdoor stages for rock concerts. The musician Fuchsia requires a square stage at all of her concerts. Rockstar Platforms lays down a stage with an area of \( x^2 + 8x + 16 \) square feet for her.

1. a. Draw a diagram to represent the area of the stage.
   b. Write expressions to represent the side lengths of Fuchsia's stage.

Fuchsia looks at the stage and says, “That's too small!” So Rockstar Platforms goes back to the drawing board and designs another square stage with an area of \( 4x^2 + 20x + 25 \) square feet.

2. What are the side lengths of Fuchsia's new stage?

The company calls Fuchsia back out to look at the stage. “I guess it will do,” she says, “but I would have preferred a stage with an area of \( 5x^2 + 12x + 4 \) square feet.” Rockstar Platforms’ foreman explains, “But then your demand for a square stage would not be met.”

3. Explain why Fuchsia's demand for a square stage would not be met if the stage had an area of \( 5x^2 + 12x + 4 \) square feet.

“Leave it like it is,” concedes Fuchsia. “Now, I need you to put up a video screen.” Fuchsia wants a large rectangular video screen set up behind her so her fans can see her from far away. Rockstar Platforms’ foreman has a plan for a video screen with an area of \( 2x^2 + 11x + 14 \). “Boss, that's not going to work,” his assistant cautions. “It's going to be too long for the stage. “It'll work,” the foreman insists.

4. Who is correct, the foreman or his assistant? Justify your response.

After finishing Fuchsia's stage, Rockstar Platforms is contracted to set up a stage for Mich.i.el. Mich.i.el wants a rectangular stage with an area of \( 2x^2 + 7x + 5 \) square feet, but he makes one very specific request. In order to fit all of his backup dancers, the length of the stage must be \( x + 4 \) feet.

5. How wide will Rockstar Platforms have to make the stage to meet Mich.i.el's request?

Two nights later, Mich.i.el and Fuchsia run into each other at a party and start arguing over who had the bigger stage.

6. How many times larger was Fuchsia's stage than Mich.i.el's stage? Write an expression in simplest form that represents the ratio of the area of Fuchsia's stage to the area of Mich.i.el's stage.
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